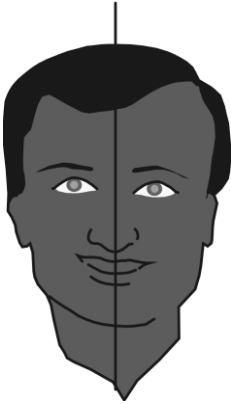


# 7. SYMMETRY GROUPS

## §7.1. What is Symmetry?

When we say that something is symmetric we're usually thinking of left-right symmetry. For example, a design is symmetric in this way if the right half is the mirror image of the left. The axis of symmetry separates the two halves and, if we place a mirror along this line, the design seems complete. The reflection of the left half makes up for the hidden right half. The human face is generally considered to have this mirror symmetry even though we can detect subtle differences between the two sides.



But this is only one type of geometric symmetry. A snowflake has more than just mirror symmetry. You can rotate it through a  $60^\circ$  angle about its centre and it fits exactly onto its original form. This is called **rotational symmetry**. The number of times this needs to be performed so that all the points return to their original positions is called the **order** or degree of the rotation. A  $60^\circ$  rotation would need to be performed 6 times to produce a full  $360^\circ$  rotation and so



has order 6. We say that a snowflake has 6-fold symmetry. Of course a  $120^\circ$  rotation rotates a snowflake to exactly the same position, even though individual points are occupying different positions. So why don't we say that it has 3-fold symmetry as well? The reason is that we quote the highest degree of symmetry which, for a snowflake, is 6. A square has 4-fold symmetry since it's fixed by a  $90^\circ$  rotation but no smaller positive angle.



The Isle of Man has a motto to the effect that “whichever way I fall, I stand” and an appropriate insignia with 3-fold symmetry, but no mirror symmetry.

A row of trees gives the appearance of translational symmetry. If all the trees are absolutely identical, and if the row extends forever in either direction, then moving each tree to the right a certain distance has no effect on the overall pattern.



Reflections, rotations and translations are the three basic operations involved in symmetry – but there are more. However let's digest these three before considering glides, screw rotations etc.

## §7.2. Symmetry Groups

A **symmetry operation**, for a subset  $X$  of the plane, is an isometry  $f$  of the plane that maps  $X$  to itself, meaning that  $f(X) = X$ . The individual points in  $X$  are not necessarily fixed but the set as a whole is. The set of all symmetry operations of  $X$  is denoted by  $\mathbf{Sym}(X)$  and is called the **symmetry group** of  $X$ .

**Example 1:** If  $X$  is the following picture of a house then  $\mathbf{Sym}(X) = \{I, \mu\}$  where  $\mu$  is the reflection in the axis of symmetry.



(If you have a careful eye you'll detect that the mirror symmetry isn't quite perfect since the chimney and door-handle don't have a corresponding chimney or door-handle on the other side of the axis. Symmetry, as employed in Western Art, often has small asymmetry, to make the image more interesting. In the art of the East imperfections in the symmetry are deliberately included because "only God can create perfection".)

**Example 2:** If  $X$  is the Isle of Man insignia then  $\text{Sym}(X) = \{I, \rho, \rho^2\}$  where  $\rho$  is the rotation about the centre through  $120^\circ$  and  $\rho^2$  is the rotation through  $240^\circ$ . The  $240^\circ$  rotation is written as  $\rho^2$  because it is equivalent to performing  $\rho$  twice in succession.

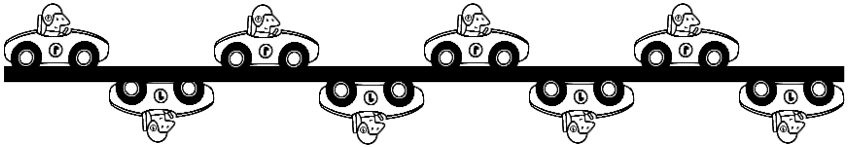


**Example 3:** If  $X$  is the following infinite pattern of equally spaced cars, arranged in a line, then:

$\text{Sym}(X) = \{\dots, \tau^{-2}, \tau^{-1}, I, \tau, \tau^2, \tau^3, \dots\}$  where  $\tau$  is the translation to the right through the distance between successive cars. So  $\tau$  takes each car to the next on the right, and so the whole infinite pattern is mapped to itself, or is fixed. The translation  $\tau^2$  moves each car two places to the right,  $\tau^{-5}$  takes each car 5 places to the left, and so on.



**Example 4:** If  $X$  is the following infinite pattern then  $\text{Sym}(X)$  contains translations, but no reflections or rotations.



However it does contain ‘glides’. A **glide** is a reflection in an axis followed by a translation parallel to that axis. In Example 4 the symmetry group contains the glide that consists of a reflection in the horizontal axis followed by a suitable translation. Each car maps to the nearest car to the right, on the opposite side of the axis. The square of this glide,  $\gamma^2$ , is the translation that sends each car to the next, on the same side of the axis. There is no mirror symmetry. The symmetry group is then generated by  $\gamma$ . That is,  $\text{Sym}(X) = \{ \dots, \gamma^{-2}, \gamma^{-1}, I, \gamma, \gamma^2, \gamma^3, \dots \}$ .

The word ‘group’ was used in the phrase ‘symmetry group’ because these symmetry groups are more than just *sets* of isometries. They have an algebraic structure. We can multiply any two symmetry operations, and if each of them fixes a certain subset of the plane then so does their product. We say that the set of symmetry operations for a given planar set is **closed** under multiplication.

A **group** is a certain type of algebraic system. It's a set on which a binary operation is defined that satisfies certain properties, called the 'group axioms'. The operation is called 'multiplication' but it doesn't have to have anything to do with multiplication of numbers. In fact the elements of the set needn't be numbers. They could be matrices or functions or points. For the sort of groups we'll be considering the elements of the set are symmetry operations, such as rotations, reflections and translations. The relevant binary operation consists of performing one symmetry operation after another (in the given order). So the product of a  $30^\circ$  rotation and a  $60^\circ$  rotation (about the same axis) is the effect of doing the  $30^\circ$  rotation and then following this by a  $60^\circ$  rotation. The net effect is a  $90^\circ$  rotation (about the same axis).

Before an algebraic system with a binary operation is allowed to be called a group it needs to satisfy the four **group axioms**.

**Closure:** The product of any two elements in the set must be again in the set.

**Associative Law:**  $A(BC) = (AB)C$ .

This holds for symmetry operations because in both cases the effect is to perform the operations A, B, C one after the other, in that order.

**Identity:** The identity operation is the operation of doing nothing. Clearly it's a symmetry operation for every set because if the individual points are fixed, the overall pattern must be too. We denote this identity by  $I$ . Note that  $IA = AI$  for all symmetry operations,  $A$ .

We could regard  $I$  as a  $0^\circ$  degree rotation, though we'd be hard-pressed to work out the centre. Or we could regard  $I$  as a translation through a distance of  $0$ , though we'd have difficulty deciding in which direction we'd moved. Instead we treat it as something special. So when we refer to a 'rotation' we exclude a  $0^\circ$  degree rotation – it would be the identity. And whenever we refer to a translation we'll never mean one that moves points through a zero distance.

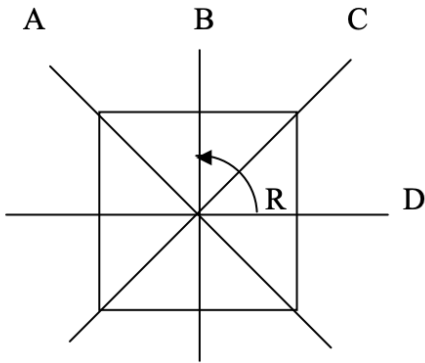
We can now write equations such as  $R^4 = I$  to indicate the fact that if a  $90^\circ$  degree rotation  $R$  is performed 4 times in succession the effect on all the points (in so far as where they end up, not what sort of journey they've been on) is to fix them (i.e. map them to themselves).

**Inverse:** In a group every element has to have an inverse relative to the operation. So for all  $A$  there must exist a  $B$  in the set such that  $AB = I = BA$ . This is true for symmetry operations because they can all be 'undone' by a symmetry operation. The inverse of a rotation through  $\theta$  is a rotation through  $-\theta$  (about the same axis). The inverse of a translation through a distance  $h$  to the right is the translation through a distance  $h$  to the left. And, of

course, the inverse of a reflection is simply the reflection itself.

### §7.3. The Symmetry Group of a Square

A square has four axes of mirror symmetry. They all run through the centre. Two of these axes run through the corners and the other two run through the midpoints of opposite sides.



Let's call the reflections about these axes A, B, C and D as in the diagram.

Then there's the  $90^\circ$  rotation anti-clockwise. (As usual, for rotations in the plane, we take anti-clockwise as the positive

direction.) Let's call this R. A  $180^\circ$  rotation can then be expressed as  $R^2$  and a  $270^\circ$  rotation (i.e.  $90^\circ$  clockwise) is  $R^3$ . Finally we have the identity, I.

These eight operations constitute the entire symmetry group of a square.

If X is a square  $\text{Sym}(X) = \{I, R, R^2, R^3, A, B, C, D\}$ . As with any finite group, we can exhibit the group table which sets out every possible product. This is the group table for the symmetry group of a square.

	<b>I</b>	<b>R</b>	<b>R<sup>2</sup></b>	<b>R<sup>3</sup></b>	<b>A</b>	<b>B</b>	<b>C</b>	<b>D</b>
<b>I</b>	I	R	R <sup>2</sup>	R <sup>3</sup>	A	B	C	D
<b>R</b>	R	R <sup>2</sup>	R <sup>3</sup>	I	B	C	D	A
<b>R<sup>2</sup></b>	R <sup>2</sup>	R <sup>3</sup>	I	R	C	D	A	B
<b>R<sup>3</sup></b>	R <sup>3</sup>	I	R	R <sup>2</sup>	D	A	B	C
<b>A</b>	A	D	C	B	I	R <sup>3</sup>	R <sup>2</sup>	R
<b>B</b>	B	A	D	C	R	I	R <sup>3</sup>	R <sup>2</sup>
<b>C</b>	C	B	A	D	R <sup>2</sup>	R	I	R <sup>3</sup>
<b>D</b>	D	C	B	A	R <sup>3</sup>	R <sup>2</sup>	R	I

You should check some of these out with a small square cut out of paper. Number the corners on one side of the paper 1, 2, 3, 4 and number the corners on the back of the paper in such a way that each corner has the same number, front or back.

Place the square so that the corners are as follows:

<b>1</b>	<b>2</b>
<b>3</b>	<b>4</b>

Now perform a pair of operations. For example take AB. Do A first and then B. It's important to remember that the axes are fixed in space. Don't write them on your paper. So, for example, B is always the vertical axis.

After doing AB look at the numbers at the corners. You should get:

<b>3</b>	<b>1</b>
<b>4</b>	<b>2</b>

Now ask yourself what single operation, from the set of eight, would

have achieved the same effect. Clearly, in this case, it's  $R^3$  so you'll have shown that  $AB = R^3$ .

Examine the table carefully. Notice, for example, that  $AB \neq BA$ . Symmetry operations frequently fail to commute. In other words we often get a different answer if we multiply in the opposite order.

You'll find that the table has a very definite pattern, with A, B, C advancing in order, either forwards or backwards as you go across the rows or down the columns. In particular note that  $RA = B$ ,  $R^2A = C$  and  $R^3A = D$ . So all eight operations can be **generated** by just R and A. Now  $R^4 = I$  and  $A^2 = I$ . Also note that  $AR = R^3A$ . We usually write this as  $AR = R^{-1}A$ . (Because  $R^4 = I$  it follows that  $R^3 = R^{-1}$ .) These relations are sufficient to compute all products in the group.

$$R^4 = I$$

$$A^2 = I$$

$$AR = RA^{-1}.$$

If we have any product of R's and A's and their powers the last relation enables us bring all the R's to the front and all the A's to the back and so we can write it as  $R^iA^j$ . The first two relations mean that we can restrict ourselves to  $i = 0, 1, 2, 3$  and  $j = 0, 1$ . Thus we can identify the product as one of the eight combinations:

$$I, R, R^2, R^3, A, RA, R^2A, R^3A.$$

These are the eight elements of the group.

For this reason we write this group in the form

$\langle R, A \mid R^4 = A^2 = I, AR = R^{-1}A \rangle$ . This means that it's the group **generated** by  $R$  and  $A$  subject to the relations specified. It is a well-known group, belonging to a family of groups, called the 'dihedral groups'.

## §7.4. Cyclic and Dihedral Groups

The simplest family of groups is the family of **cyclic** groups. These are groups with just one generator. If we just stuck to the rotations in the symmetry group of a square we'd get the group  $\{I, R, R^2, R^3\} = \langle R \mid R^4 = I \rangle$ .

A **cyclic group of order  $n$**  is a group of the form  $\langle A \mid A^n = I \rangle$ , where  $A$  is a **generator**. (Saying 'of order  $n$ ' for a group is simply saying that it has size  $n$ .)

The elements are  $I, R, R^2, \dots, R^{n-1}$ . The **infinite cyclic group** is generated by a single generator with no relations and can be expressed as  $\langle A \mid \ \rangle$ . Here the elements are not just  $I, A, A^2, A^3, \dots$  but also include the negative powers  $A^{-1}, A^{-2}, A^{-3}, \dots$

**Notation:** The cyclic group of order  $n$  is denoted by  $C_n$  and the infinite cyclic group is denoted by  $C_\infty$ .

The symmetry groups of each of the sets in Examples 1 and 2 are cyclic. The one involving the infinite pattern of cars is the infinite cyclic group.

Since any two powers of the same generator commute we can say that a cyclic group is ‘commutative’. But a more usual adjective that we use is ‘**abelian**’, in honour of the Norwegian mathematician Abel (1802 – 1829), so all cyclic groups are abelian.

The next simplest family of groups, and the place where we meet non-abelian groups for the first time, is the family of dihedral groups.

A **dihedral group of order  $2n$**  is a group of the form

$$\langle A, B \mid A^n = B^2 = I, BA = A^{-1}B \rangle.$$

The **infinite dihedral group** is

$$\langle A, B \mid B^2 = I, BA = A^{-1}B \rangle.$$

**Notation:** The dihedral group of order  $2n$  is denoted by  $\mathbf{D}_{2n}$  and the infinite dihedral group is denoted by  $\mathbf{D}_\infty$ .

**Theorem 1:** The elements of

$$\mathbf{D}_{2n} = \langle A, B \mid A^n = B^2 = I, BA = A^{-1}B \rangle \text{ are:}$$

$$I, A, A^2, \dots, A^{n-1}, B, AB, A^2B, \dots, A^{n-1}B.$$

**Proof:** Because of the relation  $BA = A^{-1}B$  any product involving powers of  $A$  and  $B$  can be written as  $A^iB^j$  for some  $i, j$ .

Since  $A^n = I$  we can limit  $i$  to the range  $0 \leq i < n$  and since  $B^2 = I$  we need only have  $j = 0, 1$ . Clearly these  $2n$  elements are distinct.

**Theorem 2:**  $\mathbf{D}_{2n}$  is abelian if and only if  $n = 1$  or  $2$ .

**Proof:** Mostly, dihedral groups are non-abelian because  $BA = A^{-1}B$ . But in  $\mathbf{D}_4$ ,  $A^2 = I$  in which case  $A = A^{-1}$  and so  $BA = AB$ . Also  $\mathbf{D}_2$  is identical to  $\mathbf{C}_2$  and so is also abelian.

**Example 5:** Simplify  $A^4BA^3BA^{-5}BA^2$  in the dihedral group  $\mathbf{D}_{12}$ :  $\langle A, B \mid A^6 = B^2 = I, BA = A^{-1}B \rangle$ .

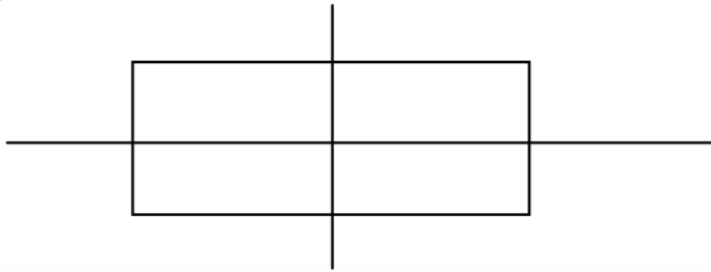
**Solution:** Using the dihedral relation we can write:

$$\begin{aligned} A^5BA^3BA^{-5}BA^2 &= A^5BA^3BA^{-5}A^{-2}B \\ &= A^5BA^3BA^{-7}B \\ &= A^5BA^3A^7B^2 \\ &= A^5BA^{10}B^2 \\ &= A^{-5}B^3 \\ &= AB, \text{ using the other relations.} \end{aligned}$$

A similar analysis to that we made for the square will apply to any regular polygon. So the symmetry group of an  $n$ -sided polygon is  $\mathbf{D}_{2n}$  where the  $A$  generator is a rotation through  $360/n^\circ$  and  $B$  is any reflection in one of the axes.

For example, if  $X$  is an equilateral triangle,  $\text{Sym}(X) \cong \mathbf{D}_6$ . There's no 2-sided polygon so  $\mathbf{D}_4$  can't arise in this way.

But the group of symmetries of a proper rectangle (one that isn't a square) is  $D_4$ .



This group is:

$\langle A, B \mid A^2 = B^2 = I, BA = AB \rangle = \{I, A, B, AB\}$  where:  
 A is a vertical reflection (in the horizontal axis),  
 B is a horizontal reflection (in the vertical axis) and  
 R = AB is the  $180^\circ$  rotation about the centre.

A group H is a **subgroup** of G if it's a subset that forms a group in its own right, under the same operation. For example the rotations in a symmetry group of a set X form a subgroup **Rot(X)**.

If H and K are subgroups of some larger group G and if:

- (1)  $hk = kh$  for all  $h \in H, k \in K$  and
  - (2) only the identity element belongs to both H and K
- then we define  $H \times K$  to be  $\{hk \mid h \in H, k \in K\}$  and we call it the direct product of H and K.

Two subgroups of some larger group are **disjoint** if

$$H \cap K = \{1\}.$$

Normally we say that two subsets are disjoint if they have no common element, but subgroups will always contain the identity and we modify the concept of disjointness.

The **direct product** of two disjoint subgroups  $H, K$  of a some larger group is:  $H \times K = \{hk \mid h \in H \text{ and } k \in K\}$ . Because  $H \cap K = \{1\}$ , it is easy to see that each element of  $H \times K$  can be expressed uniquely as  $hk$ .

**Example 6:**

$$\langle A, B, C \mid A^4 = B^2 = C^3 = I, BA = A^{-1}B, CA = AC, CB = BC \rangle$$

is the direct product of  $\langle A, B \mid A^4 = B^2 = I, BA = A^{-1}B \rangle$  and  $\langle C \mid C^3 = I \rangle$  and so may be written as  $\mathbf{D}_8 \times \mathbf{C}_3$ .

**Example 7:** Show that  $\mathbf{D}_4$  is the same group as  $\mathbf{C}_2 \times \mathbf{C}_2$ .

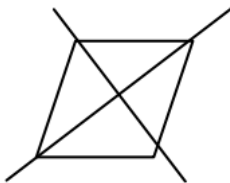
**Solution:**  $\mathbf{D}_4 = \langle A, B \mid A^2 = B^2 = I, BA = A^{-1}B \rangle$ . Since  $A^2 = I$  in this group it follows that  $A^{-1} = A$ , so  

$$\begin{aligned} \mathbf{D}_4 &= \langle A, B \mid A^2 = B^2 = I, BA = AB \rangle \\ &= \langle A \mid A^2 = I \rangle \times \langle B \mid B^2 = I \rangle \\ &= \mathbf{C}_2 \times \mathbf{C}_2. \end{aligned}$$

**Example 8:** Find the symmetry group of:

- (i) a rhombus;
- (ii) a proper parallelogram (one that isn't a rhombus or a rectangle);
- (iii) an equilateral triangle;
- (iv) an isosceles right-angled triangle.

**Solution:** (i) A rhombus has two axes of reflection as well as a  $180^\circ$  rotation about the centre. So its symmetry group is  $\mathbf{D}_4 = \mathbf{C}_2 \times \mathbf{C}_2$  just like the proper rectangle.

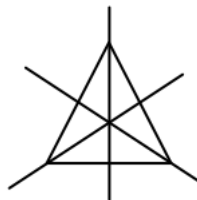


(ii) With a proper parallelogram these reflections no longer fix the figure. All we have is a  $180^\circ$  rotation and, of course, the identity. The symmetry group is thus  $\mathbf{C}_2$ .

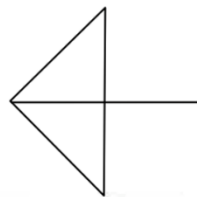
(iii) The symmetry group of an equilateral triangle is

$$\mathbf{D}_6 = \langle A, B \mid A^3 = B^2 = I, BA = A^{-1}B \rangle$$

where  $A$  is a  $120^\circ$  rotation about the centre and  $B$  is a  $180^\circ$  rotation about any of the three axes of symmetry.



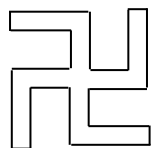
(iv) An isosceles right-angled triangle is not equilateral so we have  $\mathbf{C}_2$  as the symmetry group. But in this case the element of order 2 is a reflection instead of a  $180^\circ$  rotation.



While dihedral groups occur as the symmetry groups of many patterns (e.g. the Mercedes Benz logo has  $\mathbf{D}_6$  symmetry) some patterns have cyclic symmetric groups. These patterns have rotational symmetry but no mirror symmetry. The most famous (or



infamous) of these is the Swastika. All such patterns come in two varieties, each being the mirror image of the other.



It's a pity this ancient symbol was soiled by the Nazis because it's a symbol with a long history, yet one cannot now help feel uncomfortable in seeing it. The earliest Swastika was found on pots from Persia which are dated from about the fourth millennium BC. It has also been found Greece, India, China and Japan, generally as a good-luck charm. (The word 'swastika' is from Sanskrit meaning 'all will be well'.) Fortunately there is a left-handed and a right-handed version. The one shown here is *not* the Nazi symbol but is its mirror image. The symmetry group of either Swastika is  $C_4$ .

The insignia for the Isle of Man (a small island between England and Ireland – a part of the U.K. but with its own parliament) has  $C_3$  as its symmetry group.



## §7.5. Finite Symmetry Groups in the Plane

A symmetry group is a way of describing the type of symmetry of a set. For example the shapes that have just mirror symmetry can be considered to have the same type of symmetry. It's natural therefore to ask "what are the possible types of symmetry for subsets of the plane?" In this section we'll restrict our attention to just finite

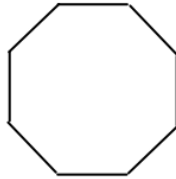
symmetry groups, so that will preclude translations and glides.

We've seen that any finite cyclic group can occur as the symmetry group of some planar set, as can any dihedral group.

**Example 9:** Find a subset of the plane that has

$$D_{16} = \langle A, B \mid A^8 = B^2 = I, BA = A^{-1}B \rangle$$

as its symmetry group.



**Solution:** Clearly an example of such a set is a regular octagon, where  $A$  is a rotation through  $360/8 = 45$  degrees and  $B$  is a reflection in any of the eight axes of symmetry.

**Example 10:** Find a subset of the plane that has  $C_7$  as its symmetry group.

**Solution:** One can modify the Isle of Man insignia by taking 7 legs, equally spaced around a circle.

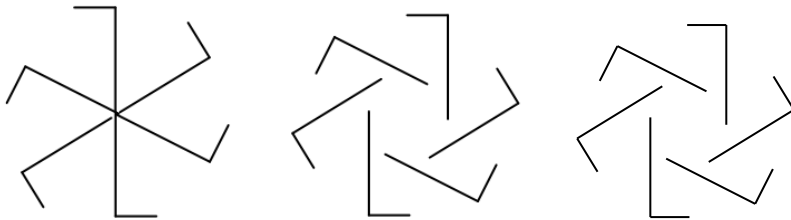
One can build up a set with symmetry group  $C_n$  by taking as a basis any shape with trivial symmetry group. We'd normally say that such a shape has no symmetry, but when talking about the symmetry group we must always include the identity.

An example of such a shape is:

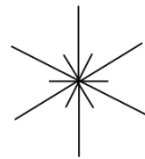


We then take any point in the plane and rotate this basic shape through various multiples of  $360/n$  degrees. The union of all these rotated copies will generally have symmetry group  $C_n$ . However particular choices may introduce further symmetry.

**Example 11:** The following sets have symmetry group  $C_6$ .



**Example 12:** Choosing to rotate each piece about the corner we get the following which has symmetry group  $D_{12}$ .



**Theorem 3:** If  $G$  is a finite group of rotations about a common point then  $G$  is cyclic.

**Proof:** Let  $R \in G$  be the rotation through the smallest positive angle  $\theta$ . Let  $S$  be any non-trivial rotation in  $G$  and

suppose it is through the positive angle  $\varphi$ . Let  $n$  be the integer part of  $\varphi/\theta$ . Then  $0 \leq \varphi - n\theta < \theta$ .

But  $S(\mathbb{R}^n)^{-1} \in G$  and it is a rotation through the angle  $\varphi - n\theta$ . If  $\varphi - n\theta > 0$  this contradicts the choice of  $R$ . Hence  $\varphi = n\theta$  and so  $S = R^n$ .

**Theorem 4:** The finite symmetry groups for subsets of the plane are cyclic or dihedral.

**Proof:** Let  $S$  be a subset of the plane with a finite symmetry group  $G$ . Take a single point in  $S$  and operate on it by all the elements of  $G$ . This gives us a finite set of points which are permuted amongst themselves by the elements of  $G$ . Call this set  $X$ .

If  $X = \{(x_1, y_1), \dots, (x_m, y_m)\}$  the centre of gravity is the point  $P = (\sum x_i/m, \sum y_i/m)$ . While this point needn't be in  $X$ , every element of  $\text{Sym}(X)$  must fix  $P$ . We can thus take  $P$  to be the origin and so  $\text{Sym}(X)$  consists entirely of central isometries.

The rotation group,  $\text{Rot}(X)$ , consists entirely of rotations about the origin and so is  $C_n = \langle R \mid R^n = I \rangle$  for some  $n$ . Any opposite isometry is a reflection in an axis through the origin. If there's no mirror symmetry  $\text{Sym}(X)$  is just  $C_n$ .

Suppose that there's mirror symmetry and let  $M \in \text{Sym}(X)$  be any reflection. Then  $\text{Sym}(X)$  is generated by  $R$  and  $M$ . If  $N$  is any other reflection then  $MN$  is a rotation. Hence  $MN = R^k$  for some  $k$  and hence:

$$N = M^{-1}R^k = MR^k.$$

It follows that  $\text{Sym}(X)$  is generated by  $R$  and  $M$ , where  $R^n = I$  and  $M^2 = I$ .

Now  $MR$  is an opposite isometry and so must be a reflection. Hence  $(MR)^2 = I$  from which we get:

$$MR = R^{-1}M.$$

Thus  $\text{Sym}(X) = \langle R, M \mid R^n = I, M^2 = I, MR = R^{-1}M \rangle$  which is isomorphic to  $D_{2n}$ .

## §7.6. The Seven Frieze Patterns:

A **frieze pattern** is any subset  $X$  of the plane whose symmetry group  $\text{Sym}(X)$  contains a translation,  $T$ , that generates all translations in  $\text{Sym}(X)$ . In other words, it's a linear pattern that repeats in one direction.

We're excluding such subsets as a single line because, although the translations have a common direction, there are translations in the Symmetry group of a line through arbitrarily small distances, none of which can generate them all.



We define two patterns to be equivalent if their symmetry groups are isomorphic in such a way that corresponding isometries have the same type (rotation, reflection etc.) The following is a list of seven frieze

patterns, no two of which are equivalent. We'll later show that these are the only ones. Every frieze pattern is equivalent to one of these seven.

We consider each of these patterns to straddle the  $x$ -axis, and we take the smallest unit of translational symmetry to be 1.

(1) ...**FFFFFFFFFFFFFFFFFFFF**...

This has no symmetry, other than translational symmetry in one direction. Its symmetry group is the infinite cyclic group,  $C_\infty = \langle T \mid \rangle$  where  $T$  is a translation to the right by one unit:  $T(x, y) = (x + 1, y)$ .

(2) ...**EEEEEEEEEEEEEEEEEE**...

This pattern has mirror symmetry in an axis parallel to the direction of the translation. Its symmetry group is  $C_\infty \times C_2 = \langle T, M \mid M^2 = I, MT = TM \rangle$  where  $T$  is as above and  $M$  is the reflection in the  $x$ -axis.

Since  $M(x, y) = (x, -y)$  it is easy to check that  $MT = TM$ .

(3) ...**AAAAAAAAAAAAAAAAAA**...

Like the pattern of  $E$ 's, above, this has mirror symmetry, as well as the translation symmetry. But it has infinitely many axes of mirror symmetry, not just one. Its symmetry group is:

$$D_\infty = \langle T, M \mid M^2 = I, MT = T^{-1}M \rangle$$

where  $T$  is as above and  $M$  is a reflection in the  $y$ -axis (if we allow one of the  $A$ 's to straddle the  $y$ -axis).

Since  $M(x, y) = (-x, y)$  it is easy to check that  $MT = T^{-1}M$ .

There is also a mirror axis at  $x = \frac{1}{2}$  (if we take the unit of translation to be 1). If  $F$  is the reflection in  $x = \frac{1}{2}$  then  $F(x, y) = (1 - x, y)$ . But this is just  $T^{-1}M$ .

(4) **···pbpbpbpbpbpbpb···**

This pattern has no mirror symmetry but it does have glide symmetry. The axis of the glide is the centre line. A reflection in this line, followed by a translation through half a unit to the right, is a glide which fixes the whole pattern. Every other glide is the product of this one and a suitable translation.

The symmetry group for this pattern is also the infinite cyclic group  $C_\infty = \langle G \mid \rangle$  where:

$$G(x, y) = (x + \frac{1}{2}, -y).$$

Every translation in the symmetry group is an even power of this glide while the odd powers are other glides.

(5) **···NNNNNNNNNNNNNN···**

This pattern has no mirror or glide symmetry. But it has rotational symmetry about the centre of each letter  $N$ . If we pick one of these rotations, the others can be obtained by multiplying it by a suitable translation. So the symmetry group is again the infinite dihedral group:

$$D_\infty = \langle T, R \mid R^2 = I, RT = T^{-1}R \rangle$$



$G(x, y) = (x + \frac{1}{2}, -y)$  and the translations are generated by  $T(x, y) = (x + 1, y)$ . Clearly  $G^2 = T$ .

If we take the origin at one of the centres of 2-fold rotational symmetry then the corresponding  $180^\circ$  rotation is  $R(x, y) = (-x, -y)$ . Let  $M$  be reflection in the line  $x = \frac{1}{4}$ , one of the axes of reflectional symmetry.

Then  $M(x, y) = (\frac{1}{2} - x, y)$ .

It's easy to check that  $RM = G$ , so we can generate the symmetry group with just  $R$  and  $M$  and so the symmetry group is:  $\langle R, M \mid R^2 = M^2 = 1 \rangle$ . There are no other relations between  $R$  and  $M$ . But if we let  $S = RM$  then the group is  $\langle S, R \mid R^2 = 1, RS = S^{-1}R \rangle$  which is  $D_\infty$ .

**Theorem 5:** Every frieze pattern is equivalent to one of the above.

**Proof:** Let  $X$  be a frieze pattern where  $\text{Sym}(X)$  contains a translation  $T$  along the horizontal axis.

The only possible rotational symmetries are 2-fold rotations. The frieze type is determined by the existence or non-existence of the following symmetry features:

$V$  = a vertical reflection (in the horizontal axis);

$H$  = a horizontal axis (in a vertical axis);

$R$  = a 2-fold rotation;

$G$  = a horizontal glide.

Thus there are at most  $2^4 = 16$  possible types of frieze symmetry. But certain combinations are impossible.

**H  $\Rightarrow$  G:** If  $\text{Sym}(X)$  contains the reflection in the horizontal axis it also contains a glide.

**RG  $\Rightarrow$  H:** The product of a 2-fold rotation and a horizontal glide fixes all horizontal lines, so it must be a horizontal reflection (in a vertical axis).

**GV  $\Rightarrow$  R:** The product of a horizontal glide and a vertical reflection fixes a point. Being a direct isometry it must be a rotation.

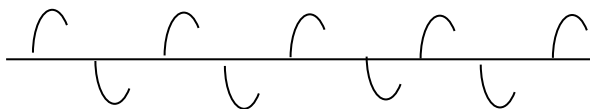
**RV  $\Rightarrow$  G:** The product of a rotation and a vertical reflection fixes only one horizontal line (the horizontal axis). It is therefore either a horizontal reflection or a horizontal glide. In either case  $\text{Sym}(X)$  contains a glide.

It follows that only the following combinations are possible:

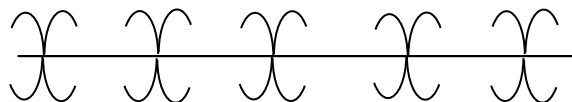
<b>combination</b>	<b>frieze type</b>
none	F
V	A
R	N
G	pb
HG	E
VG	MW
VRGH	H

**Example 13:** Classify the following frieze patterns:

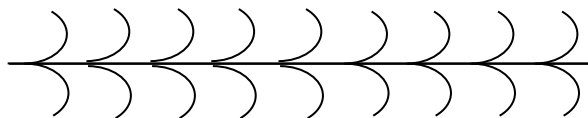
(a)



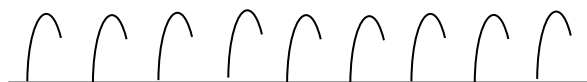
(b)



(c)



(d)



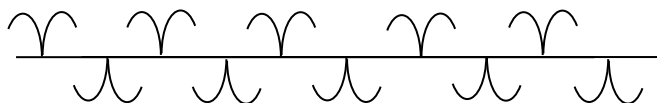
(e)



(f)



(g)



**Solution:** (a) pb; (b) H; (c) E; (d) F; (e) N; (f) A; (g) MW.

**Example 14:** Classify the graphs of the following functions as frieze patterns:

(a)  $\sin x$ ; (b)  $\cos x$ ; (c)  $\tan x$ ; (d)  $|\sin x|$ .

**Solution:** (a) MW; (b) MW; (c) N; (d) A.

## §7.7. The 17 Wallpaper Patterns

A **wallpaper pattern** is any subset,  $X$ , of the plane whose translational symmetry group  $\text{Sym}(X)$  contains translations,  $S$ ,  $T$ , in two different directions, which together generate all the translations in  $\text{Sym}(X)$ . In other words they're patterns that cover the whole plane and are repetitive in two independent directions.

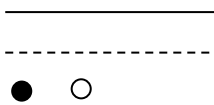
There are 17 different types of wallpaper patterns. We won't provide a proof, but we do describe these 17 types. There are several different naming conventions for them. We'll use three parameters to describe each one.

### **xyz will mean that:**

$x$  is the maximal rotational symmetry about any point;

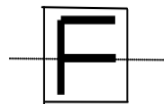
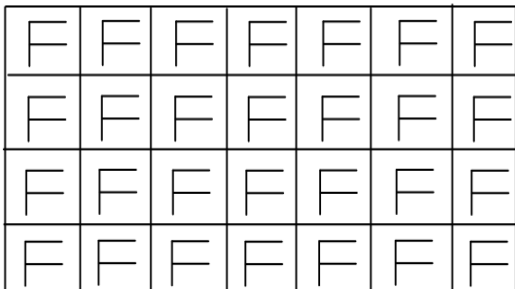
$y$  is the maximum number of mirror axes through any point;

$z$  is the maximum number of glide axes through any point.

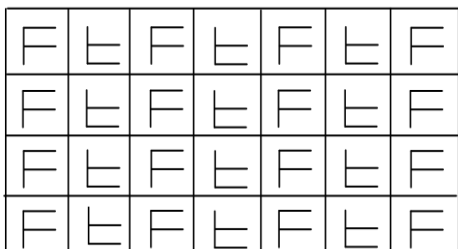


represents a mirror axis  
 represents a glide axis  
 represent centres of rotational symmetry

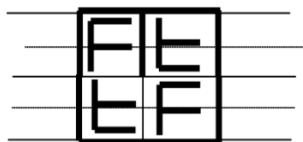
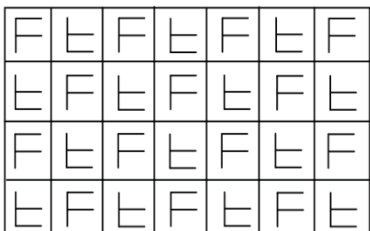
**100:**



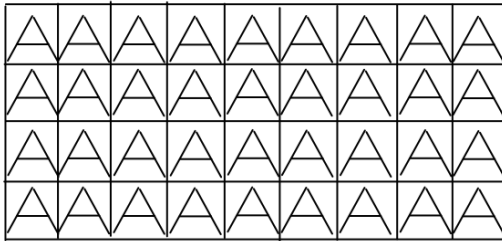
**101:**



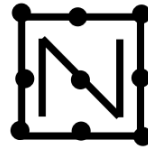
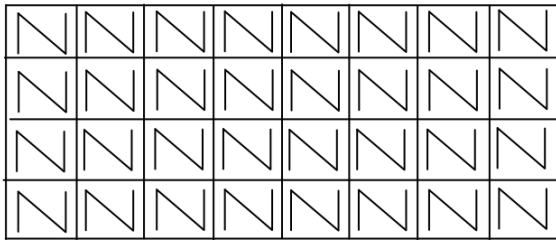
**111:**



**110:**

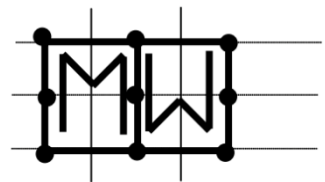
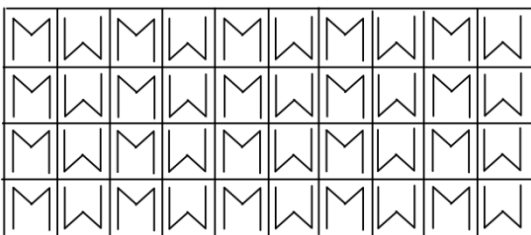


**200:**

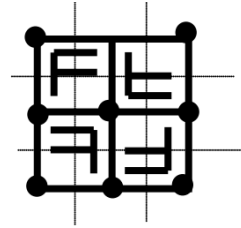
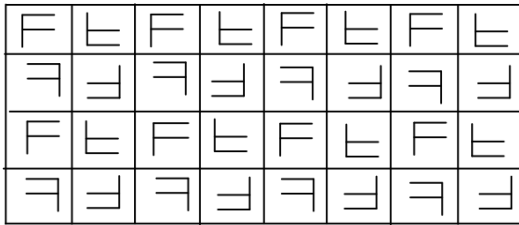


Here we have 2-fold rotational symmetry but no reflectional or glide symmetry

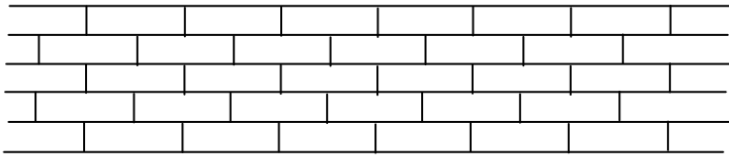
**202:**



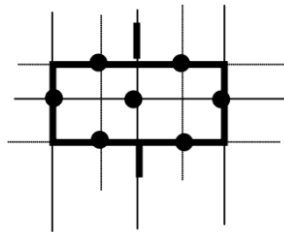
211:



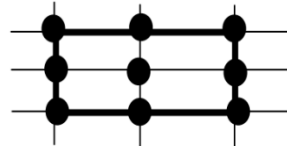
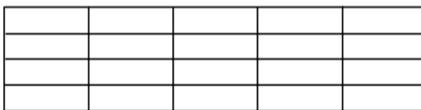
222:



There are mirror and glide axes in both directions as well as 2-fold poles.

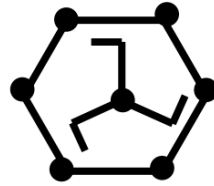
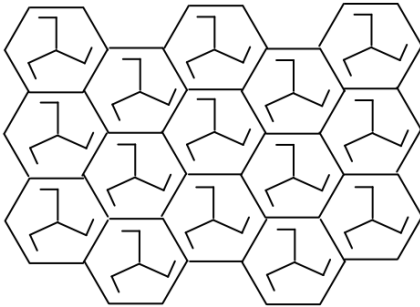


220:



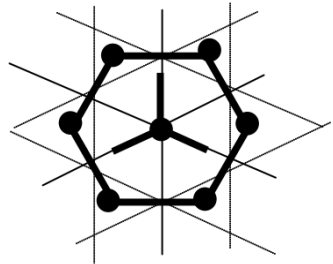
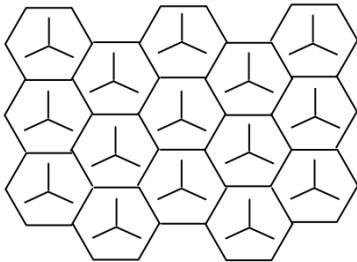
Here there are 2-fold poles and horizontal and vertical mirror symmetry. But there are no glides.

**300:**



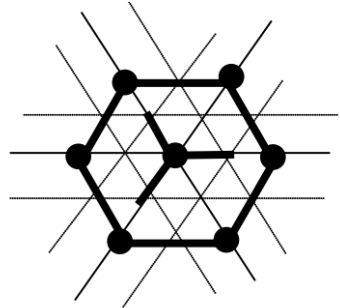
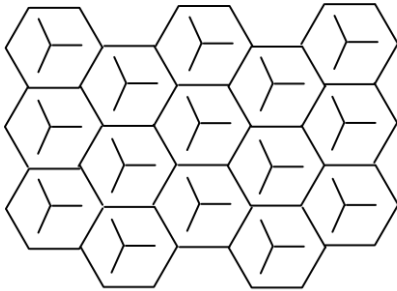
This pattern has 3-fold rotational symmetry but, because of the asymmetry of the blades of the ‘propeller’, there is no reflectional or glide symmetry.

**332A:**



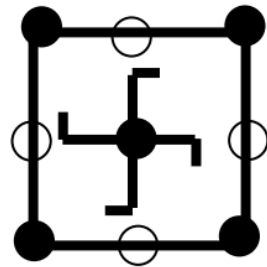
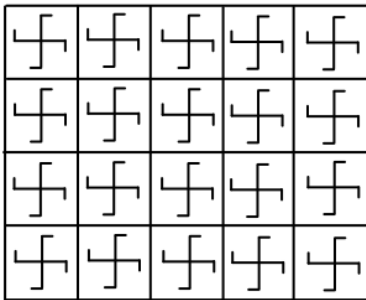
Here we have 3-fold rotational symmetry together with both reflectional and glide symmetry.

**332B:**



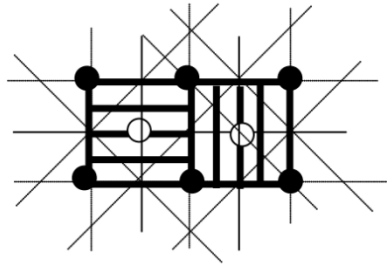
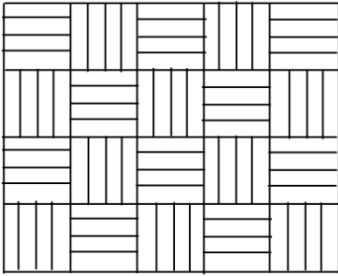
Here there's 3-fold rotational symmetry as well as both reflectional and glide symmetry. But unlike the previous pattern, all the poles lie on mirror axes.

**400:**



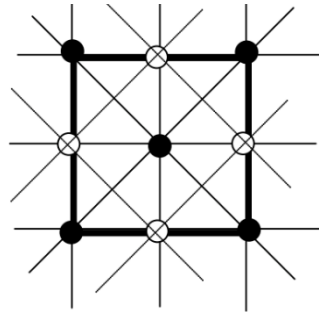
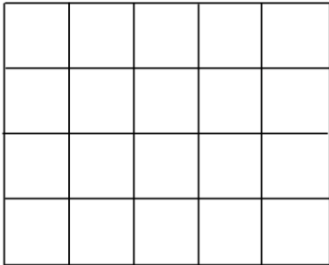
In this pattern there are some poles with 4-fold symmetry and others with only 2-fold symmetry. There is no reflectional or glide symmetry.

**423:**

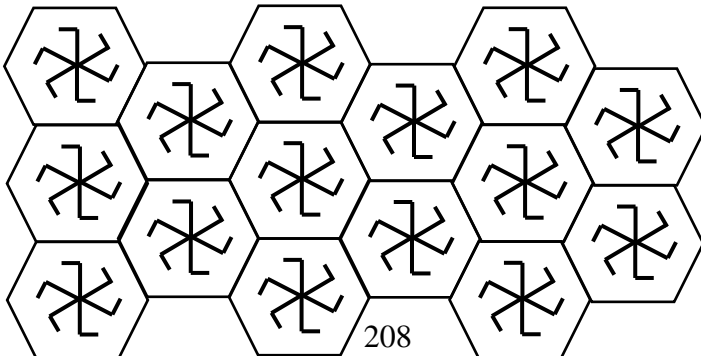


For this pattern there are both 2-fold and 4-fold poles as well as rotational and glide symmetry in both directions.

**442:**

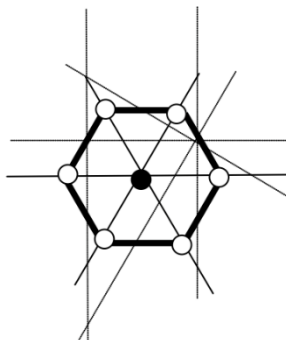
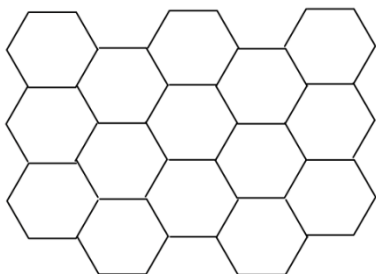


**600:**



**I've left the identification of poles and axes of mirror and glide symmetry as an exercise.**

**664:**



Here, for clarity, I've only shown the glide axes through a single point.

## EXERCISES FOR CHAPTER 7

**Exercise 1:** Draw a diagram to show the poles (centres of rotational symmetry and the axes of mirror and glide symmetry) for wallpaper pattern 600.

### Exercise 2:

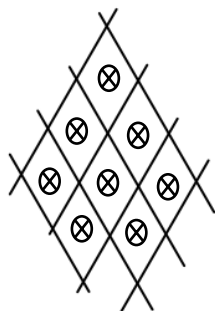
Describe the following wallpaper patterns by the code  $xyz$  where:

$x$  = maximum rotational symmetry about any point;

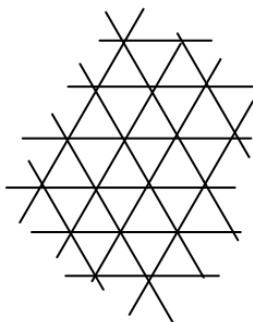
$y$  = the maximum number of mirror axes through any point;

$z$  = the maximum number of (proper) glide axes through any point;

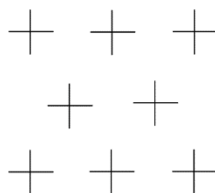
(a)



(b)



(c)



### Exercise 3:

A certain room has exposed brick walls, a wooden parquet floor, square acoustic tiles in the ceiling and diamond-pane leaded glass windows (each pane is a rhombus).

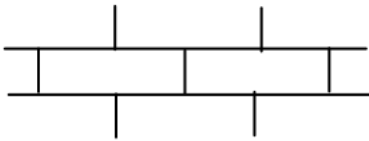
Write down the symbol  $xyz$  for each of these patterns (assuming each is a portion of a complete wall-papering of the plane) where:

$x$  is the maximum degree of rotational symmetry about any point,

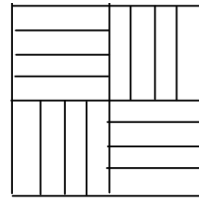
$y$  is the maximum number of mirror axes through any point and

$z$  is the maximum number of proper glide axes through any point.

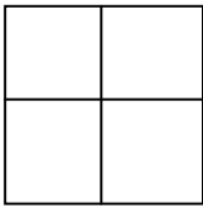
(a) bricks



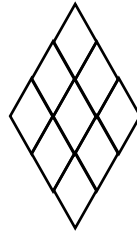
(b) parquet flooring



(c) ceiling tiles

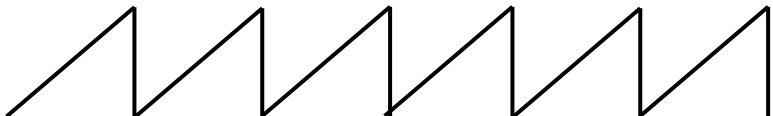


(d) window panes



**Exercise 4:** Identify the rotational, mirror and glide symmetry for each of the following frieze patterns:

(a)



(b)

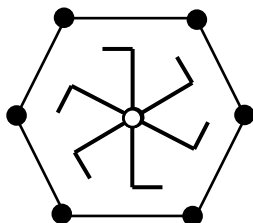


(c)



# SOLUTIONS FOR CHAPTER 7

## Exercise 1:



There are two types of poles, ones with 3-fold symmetry (the black circles) and ones with 6-fold symmetry (the white circles). There are no axes of mirror or glide symmetry because on a reflection or glide the ‘Catherine Wheel motifs’ will spin in the opposite direction (imagine the short sections are rockets).

**Exercise 2:** (i) **222**, (ii) **664**; (iii) **222**.

## Exercise 3:

- (a) bricks: 222; (b) parquet floor: 423;  
(c) ceiling tiles: 442; (d) window panes: 222.

## Exercise 4:

(a) **Rotational symmetry:** This has 2-fold rotational symmetry about the midpoint of each line.

**Mirror Symmetry:** none

**Glide Symmetry:** none

(b) **Rotational symmetry:** This has 2-fold rotational symmetry about the points on the horizontal axis midway between successive scrolls.

**Mirror Symmetry:** This has vertical axes of symmetry through the middle of each scroll.

**Glide symmetry:** This has a horizontal axis of glide symmetry. If the scrolls on each side of the line are 1 unit apart, reflecting the pattern in the horizontal axis and translating to the right through a distance of  $\frac{1}{2}$  fixes the pattern.

(c) **Rotational symmetry:** This has 2-fold rotational symmetry about the midpoints of each H and about the points midway between two successive H's

**Mirror Symmetry:** This has a horizontal axis of symmetry and infinitely many vertical axes of symmetry. The vertical axes go through the centres of rotational symmetry.

**Glide Symmetry:** There are no proper glides in the symmetry group of this pattern.