7. LINEAR ODEs WITH CONSTANT COEFFICIENTS

§7.1. Higher Order ODEs

An \( n \)th order ODE involves derivatives up to \( \frac{d^n y}{d x^n} \). The context in which to discuss the solutions to higher order ODEs is abstract vector spaces. You may not yet have studied these so here’s a brief account that will be sufficient for our purposes here.

We know that vectors are things like \((x, y, z)\) that can represent points in space. You may not yet know that functions of a real variable can be viewed as functions. What is needed for vectors is a way of adding two together and multiplying one of them by a scalar. In this case the scalars will be the real numbers. There is a whole bunch of axioms, but we won’t bother discussing them here.

The functions \( f(x) = \sin x \) and \( g(x) = \cos x \) can be considered as vectors. We can add them: \( f(x) + g(x) = \sin x + \cos x \). We can also multiply them by real numbers: \( 2f(x) = 2\sin x \). But don’t vectors have to have components? Not really, but you could think of the values of the function as its components, in which case we have infinitely many!

A linear combination of the functions \( f_1(x), f_2(x), ..., f_n(x) \) is a function of the form
\[
a_1f_1(x) + a_2f_2(x) + ... + a_nf_n(x).
\]

**Example 1:** \( 3\sin x - 5\cos x \) is a linear combination of \( \sin x \) and \( \cos x \).

**Example 2:** The constant function \( f(x) = 1 \) is a linear combination of the functions \( g(x) = \sin^2 x \) and \( \cos^2 x \) since \( \sin^2 x + \cos^2 x = 1 \).

The zero function \( z(x) = 0 \) is a linear combination of the functions \( f(x) = 1 \), \( g(x) = \sin^2 x \) and \( h(x) = \cos^2 x \) since \((-1)1 + 1\sin^2 x + 1\cos^2 x = 0 \).

Of course the zero function \( z(x) = 0 \) is always a linear combination of any set of functions. For example. \( 0.1 + 0.\sin^2 x + 0.\cos^2 x = 0 \).

But we call such a linear combination, where all the coefficients are zero, a trivial linear combination. If there’s a non-trivial linear combinations of \( f_1(x), f_2(x), ..., f_n(x) \) we say that the functions \( f_1(x), f_2(x), ..., f_n(x) \) are linearly dependent. Any \( f_i(x) \) with a non-zero coefficient can be expressed as a linear combination of the others.

**Example 3:** The functions \( \sin^2 x, \cos^2 x \) and the constant function \( 1 \) are linearly dependent since \( \sin^2 x + \cos^2 x - 1 = 0 \). We can therefore write \( \cos^2 x = 1 - \sin^2 x \).

If \( f_1(x), f_2(x), ..., f_n(x) \) are not linearly dependent we say that they are linearly independent. In that case none of the functions are a linear combination of the others.
Example 4: Show that the following functions are linearly independent:
\[ f(x) = \sin x, \quad g(x) = \cos x, \quad h(x) = x. \]

Solution: It’s no good saying “I can’t think of any connection between these three functions.” A proof is required. We do this as a Proof By Contradiction.

Suppose \( a \sin x + b \cos x + cx = 0 \) for some real numbers \( a, b, c \).

Now this is not an equation to be solved for certain \( x \). It is a relationship between three functions and therefore holds for all \( x \).

Put \( x = 0 \). Then \( b = 0 \).

Put \( x = \pi \). Then \( c\pi = 0 \), so \( c = 0 \).

Put \( x = \pi/2 \). Then \( a = 0 \).

So there are no non-trivial linear combinations that equal the zero function and hence these functions are linearly independent.

§7.2. Linear ODEs With Constant Coefficients

An \( n \)th order linear ODE with constant coefficients is one of the form

\[
\frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = Q(x) \quad \text{where the} \quad a_i \text{'s are constants.}
\]

The associated homogeneous ODE is

\[
\frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = 0.
\]

There are two fundamental facts about linear ODEs with constant coefficients. We state them here without proof.

Theorem 1: (1) The general solution to a linear ODE with constant coefficients has the form

\[ y = H(x) + P(x) \]

where

- \( H(x) \) is the general solution to the associated homogeneous ODE and
- \( P(x) \) is any particular solution to the original ODE.

(2) There are \( n \) linearly independent solutions to a homogeneous \( n \)th order linear ODE, but no more.

This means that if we have to solve the ODE

\[
\frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = Q(x).
\]

we find just one solution, \( g(x) \) to this equation and \( n \) linearly independent solutions

\( f_1(x), f_2(x), \ldots, f_n(x) \)

to the associated homogeneous ODE and the general solution to the original ODE will be

\[ y = A_1 f_1(x) + A_2 f_2(x) + \ldots + A_n f_n(x) + g(x). \]

There will be as many arbitrary constants in the solution as the order of the ODE.

§7.3. The Homogeneous Case

We define the characteristic equation of the homogeneous linear ODE with constant coefficients

\[
\frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = 0 \quad \text{to be the polynomial} \quad \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0
\]

Here the symbol \( \lambda \) is an indeterminate and represents a complex number.
**Theorem 2:** Suppose \( \lambda \) is a real solution to the characteristic equation. Then \( y = e^{\lambda x} \) is a solution to the homogeneous ODE.

**Proof:** Let \( y = e^{\lambda x} \).

Then \( \frac{dy}{dx} = \lambda e^{\lambda x} \),
\[ \frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x} \]

\[ \cdots \cdots \]
\[ \frac{d^n y}{dx^n} = \lambda^n e^{\lambda x} . \]

Hence \( \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = (\lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0) e^{\lambda x} = 0. \)

**Example 5:** Solve the ODE \( \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} = 21y \) if \( \frac{dy}{dx} = -5 \) and \( y = 5 \) when \( x = 0 \).

**Solution:** Writing the ODE as \( \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} - 21y = 0 \), the characteristic equation is \( \lambda^2 + 3\lambda - 21 = 0. \)

This factorises as \((\lambda - 3)(\lambda + 7) = 0\), giving solutions \( \lambda = 3, -7 \).

Hence the general solution is \( y = A e^{3x} + B e^{-7x} \).

Hence \( \frac{dy}{dx} = 3A e^{3x} - 7B e^{-7x} \).

Using the initial conditions we get
\[
\begin{aligned}
A + B &= 5 \\
3A - 7B &= -5
\end{aligned}
\]

Solving, we get \( A = 3 \) and \( B = 2 \).

Hence the solution is \( y = 3 e^{3x} + 2 e^{-7x} \).

This leaves the case of repeated solutions and non-real solutions to the characteristic equation. We set out the full story in the following theorem but I don’t intend to provide a proof.

**Theorem 3:** Suppose the characteristic equation of an \( n \)th order homogeneous linear ODE (with constant coefficients) has the following solutions:

**Real Solutions:** \( \lambda_1, \ldots, \lambda_r \), with \( \lambda_k \) having multiplicity \( m_k \) PLUS

**Non-Real Solutions:** \( \alpha_1 \pm \beta_1 i, \ldots, \alpha_s \pm \beta_s i \), with \( \alpha_k \pm \beta_k i \) having multiplicity \( n_k \).

(So that \( r + 2s = n \).)

Then the homogeneous ODE has the general solution
\[
y = \sum_{k=1}^{r} A_k(x) e^{\lambda_k x} + \sum_{k=1}^{s} e^{\alpha_k x} \left[ B_k(x) \cos(\beta_k x) + C_k(x) \sin(\beta_k x) \right].
\]

where \( A_k(x) \) is an arbitrary polynomial of degree \( r_k - 1 \) and \( B_k(x) \) and \( C_k(x) \) are arbitrary polynomials of degree \( s_k - 1 \).
Example 6: Solve the ODE \( \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0 \) if \( \frac{dy}{dx} = -7 \) and \( y = 9 \) when \( x = 0 \).

Solution: The characteristic equation is \( \lambda^2 - 2\lambda + 5 = 0 \).

Using the quadratic equation formula we get \( \lambda = -1 \pm 2i \).

Hence the general solution is \( y = e^{-x}[A\cos 2x + B\sin 2x] \).

Example 7: Find the general solution to a linear ODE with constant coefficients if the characteristic equation has solutions: 3, 3, 3, 5, 2 \( \pm \) 3i, 2 \( \pm \) 3i.

Solution: The general solution will be:

\[ y = (A + Bx + Cx^2)e^{3x} + De^{5x} + e^{2x}[(E + Fx) \cos 3x + (G + Hx) \sin 3x] \]

§7.4. Finding Particular Solutions

There are many methods given for finding a particular solution to a non-homogeneous ODE (linear with constant coefficients)

\[ \frac{d^n y}{dx^n} + a_n \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = Q(x) \]

None of them works in all cases but they all provide a particular solution when \( Q(x) \) has terms that are polynomials, exponential functions, the sine and cosine functions and products of these.

The simplest method is a sophisticated form of ‘trial and error’. Take each term in \( Q(x) \) and differentiate it repeatedly. If you get to a stage where the original term, and its successive derivatives are linearly dependent, you are in luck. You simply take a general linear combination of these functions. You then substitute it into the ODE and find the correct coefficients in the linear combination to get a solution. This probably sounds very vague. Indeed it is only when you have seen many examples that you will come to fully understand the technique.

Example 8: Find a particular solution to the ODE \( \frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = -2x^2 \).

Solution: Differentiating \(-2x^2\) gives \(-4x\). The second derivative will be \(-4\). The third derivative is 0 and we have reached linear dependence because

\[ 1 \left( \frac{d^3 y}{dx^3} \right) + 0, \quad 1 \left( \frac{d^2 y}{dx^2} \right) + 0, \quad \left( \frac{dy}{dx} \right) + 0, y = 0 \]

is a non-trivial linear combination that is zero.

So we take as a ‘trial’ solution a general quadratic \( y = ax^2 + bx + c \).

Then \( \frac{dy}{dx} = 2ax + b \) and \( \frac{d^2 y}{dx^2} = 2a \).

Substituting into the ODE we get:
(2a) + 5(2ax + b) − 6(ax^2 + bx + c) = −2x^2.

∴ (6a − 2)x^2 + (6b − 10a)x + (6c − 5b − 2a) = 0.

Since the functions x^2, x and 1 are linearly independent (easily shown) this must be the trivial linear combination, and so

\[
\begin{align*}
6a - 2 &= 0 \\
6b - 10a &= 0 \\
6c - 5b - 2a &= 0
\end{align*}
\]

Solving this system we get \(a = \frac{1}{3}, \ b = \frac{5}{9}, \ c = \frac{31}{54}\).

Hence a particular solution is \(y = \frac{1}{3}x^2 + \frac{5}{9}x + \frac{31}{54}\).

**Example 9:** Find the complete solution to the ODE \(\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = -2x^2\) if

\(y = 0\) and \(\frac{dy}{dx} = \frac{1}{2}\) when \(x = 0\).

**Solution:** The associated homogeneous ODE is \(\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0\), which has characteristic equation \(\lambda^2 + 5\lambda - 6 = 0\).

∴ \((\lambda + 6)(\lambda - 1) = 0\) and so \(\lambda = -6\) or 1.

The general solution to the homogeneous ODE is \(y = Ae^{-6x} + Be^x\) and so the general solution to the non-homogeneous one is \(y = Ae^{-6x} + Be^x + \frac{1}{3}x^2 + \frac{5}{9}x + \frac{31}{54}\).

\[\frac{dy}{dx} = -6Ae^{-6x} + Be^x + \frac{2}{3}x + \frac{5}{9}.\]

When \(x = 0\), \(y = A + B + \frac{31}{54}\) and \(\frac{dy}{dx} = -6A + B + \frac{5}{9}\).

Hence:

\[
\begin{align*}
A + B + \frac{31}{54} &= 0 \\
-6A + B + \frac{5}{9} &= \frac{1}{2}
\end{align*}
\]

This gives \(A = -\frac{2}{27}\) and \(B = -\frac{1}{2}\).

The solution is thus \(y = -\frac{2}{27}e^{-6x} - \frac{1}{2}e^x + \frac{1}{3}x^2 + \frac{5}{9}x + \frac{31}{54}\)

\[= \frac{1}{54} \left[-4e^{-6x} - 27e^x + 18x^2 + 30x + 31\right].\]

**Example 10:** Find a particular solution to the ODE \(\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 16e^{2x}\).

**Solution:** The derivative of \(16e^{2x}\) is \(32e^{2x}\) so we get linear dependence.

So our trial solution should be \(y = ae^{2x}\).

\[\frac{dy}{dx} = 2ae^{2x} \text{ and } \frac{d^2y}{dx^2} = 4ae^{2x}.\]

Substituting into the ODE we get

\[4ae^{2x} + 10ae^{2x} - 6ae^{2x} = 16e^{2x}\]

so \(8a = 16\) which gives \(a = 2\). Hence \(y = 2e^{2x}\) is a particular solution.
Example 11: Find a particular solution to the ODE $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 2e^{-6x}$.

Solution: The derivative of $2e^{-6x}$ is $-12e^{-6x}$ so we get linear dependence.
So our trial solution should be $y = ae^{-6x}$.

\[ \frac{dy}{dx} = -6ae^{-6x} \quad \text{and} \quad \frac{d^2y}{dx^2} = 36a e^{-6x}. \]

Substituting into the ODE we get
\[ 36a e^{-6x} - 30ae^{-6x} - 6ae^{-6x} = 2e^{-6x} \text{ so } 0a = 2 \text{ which has no solution.} \]

This does not mean that the ODE has no solution, simply that we took the wrong ‘trial’ solution. The reason is that the general solution to the associated homogeneous ODE is
\[ y = Ae^{-6x} + Be^x \]
and so adding another $e^{-6x}$ term will still give $0$ on the right hand side.

Without going into why, let me simply say if there is overlap between the homogeneous solution and the trial solution, multiply the trial solution by $x$ until you get a term that is not covered by the homogeneous solution. In this case we try $y = ax e^{-6x}$.

\[ \frac{dy}{dx} = ae^{-6x} - 6ax e^{-6x} \quad \text{and} \quad \frac{d^2y}{dx^2} = -6ae^{-6x} - 6ae^{-6x} + 36axe^{-6x} \]
\[ = -12a e^{-6x} + 36ax e^{-6x}. \]

Substituting, we get
\[ (-12a e^{-6x} + 36ax e^{-6x}) + 5(a e^{-6x} - 6ax e^{-6x}) - 6ae^{-6x} = 2e^{-6x} \]
\[ \therefore a e^{-6x} (-12 + 5) = 2e^{-6x} \]
\[ \therefore a = -2. \]

Hence \( y = -\frac{2}{7} x e^{-6x} \) is a particular solution.

Example 12: Find a particular solution to the ODE $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = -60\sin 2x$.

Solution: The derivative of $7\sin 2x$ is $14\cos x$ and the second derivative is $-28\sin 2x$.
So we’ve reached linear dependence. So we try $y = a\sin 2x + b\cos 2x$.

\[ \frac{dy}{dx} = 2a\cos 2x - 2b\sin 2x. \]
\[ \frac{d^2y}{dx^2} = -4a\sin 2x - 4b\cos 2x. \]

Substituting, we get
\[ (-4a\sin 2x - 4b\cos 2x) + 5(2a\cos 2x - 2b\sin 2x) - 6(a\sin 2x + b\cos 2x) = -60\sin 2x \]
\[ \therefore (-4a - 10b - 6a)\sin 2x + (-4b + 10a - 6b)\cos 2x = -60\sin 2x. \]
\[ \therefore 10a + 10b = 60 \text{ and } 10a - 10b = 0. \]
\[ \therefore a = b \text{ and } 20a = 60, \text{ so } a = b = 3. \]

The particular solution is therefore \( y = 3\sin 2x + 3\cos 2x. \)
Example 13: Find a particular solution to the ODE \( \frac{d^2y}{dx^2} + y = \sin x \).

Solution: The derivative of \( \sin x \) is \( \cos x \) and the second derivative is \( -\sin x \). So we have reached linear dependence. So we might try \( y = a \sin x + b \cos x \). But wait. We must first consider the homogeneous solution to see if there’s any overlap.

The characteristic equation is \( \lambda^2 + 1 = 0 \), which has solutions \( \pm i \). This means that the general solution to the associated homogeneous solution is \( y = A \sin x + B \cos x \). This overlaps the right hand side of our non-homogeneous equation and so we must multiply by \( x \).

So we try \( y = a x \sin x + b x \cos x \).

\[
\begin{align*}
\therefore \frac{dy}{dx} &= a \sin x + a x \cos x + b \cos x - b x \sin x, \\
\therefore \frac{d^2y}{dx^2} &= a \cos x + a (\cos x - x \sin x) - b \sin x - b (\sin x + x \cos x) \\
&= 2a \cos x - 2b \sin x - a x \sin x - b x \cos x
\end{align*}
\]

Substituting, we get
\[
2a \cos x - 2b \sin x - a x \sin x - b x \cos x = \sin x.
\]

\[
\therefore 2a \cos x - 2b \sin x = \sin x.
\]

\[
\therefore a = 0 \text{ and } b = -\frac{1}{2}.
\]

The particular solution is \( y = -\frac{1}{2} \sin x \).

Example 14: Find a particular solution to the ODE \( \frac{d^2y}{dx^2} - 6 \frac{dy}{dx} + 9y = e^{3x} \).

Solution: The derivative of \( e^{3x} \) is \( 3e^{3x} \) and so we have reached linear dependence. Our first thought is to try \( y = a e^{-6x} \).

The characteristic equation of the associated homogeneous solution is \( \lambda^2 - 6\lambda + 9 = 0 \), which has repeated solutions \( \lambda = 3, 3 \). This means that the general solution to the associated homogeneous solution is \( y = Ae^x + Bxe^x \). This overlaps the right hand side of our non-homogeneous equation and so we must multiply by \( x^2 \). (Clearly just multiplying by \( x \) is not sufficient.)

So we try \( y = ax^2 e^{3x} \).

\[
\begin{align*}
\therefore \frac{dy}{dx} &= 2ax e^{3x} + 3ax^2 e^{3x}, \\
\therefore \frac{d^2y}{dx^2} &= 2a e^{3x} + 6ax e^{3x} + 6axe^{3x} + 9ax^2 e^{3x} \\
&= 2ae^{3x} + 12axe^{3x} + 9ax^2 e^{3x}.
\end{align*}
\]

Substituting, we get
\[
(2ae^{3x} + 12ax e^{3x} + 9ax^2 e^{3x}) - 6(2axe^{3x} + 3ax^2 e^{3x}) + 9ax^2 e^{3x} = e^{3x}.
\]

\[
\therefore 2ae^{3x} = e^{3x}.
\]

\[
\therefore a = \frac{1}{2} \text{ and so the particular solution is } y = \frac{1}{2}x^2 e^{3x}.
\]
SUMMARY OF LINEAR ODEs WITH CONSTANT COEFFICIENTS

HOMOGENEOUS: \( \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = 0 \) (\( a_k \)'s are constants).

Solve the Characteristic Equation: \( \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0 \).

Real Solution \( \lambda \) with multiplicity \( m \): \( (A_1 + A_2 x + \ldots + A_{m-1} x^{m-1}) e^{\lambda x} \)

Non-Real Solution \( \alpha \pm \beta i \) with multiplicity \( m \):
\( (A_1 + A_2 x + \ldots + A_{m-1} x^{m-1}) \cos \beta x + (B_1 + B_2 x + \ldots + B_{m-1} x^{m-1}) \sin \beta x \)

Add these together for the various solutions.

NON-HOMOGENEOUS: \( \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = Q(x) \).

<table>
<thead>
<tr>
<th>General Solution to Non-Homogeneous</th>
<th>=</th>
<th>General Solution to Homogeneous</th>
<th>+</th>
<th>Particular Solution to Non-Homogeneous</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q(x)</td>
<td>BASIC TRIAL SOLUTION</td>
<td>p(x)</td>
<td>general polynomial of degree ( n )</td>
<td>e^{\alpha x}</td>
</tr>
<tr>
<td>sin \alpha x</td>
<td>( a \sin \alpha x + b \cos \alpha x )</td>
<td>cos \alpha x</td>
<td>( a \sin \alpha x + b \cos \alpha x )</td>
<td>p(x) e^{\alpha x}</td>
</tr>
<tr>
<td>p(x) sin \alpha x</td>
<td>( a(x) \sin x + b(x) \cos x ) where ( a(x), b(x) ) are polynomials of degree ( n )</td>
<td>p(x) cos \alpha x</td>
<td>( a(x) \sin x + b(x) \cos x ) where ( a(x), b(x) ) are polynomials of degree ( n )</td>
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If a term in the trial solution overlaps with a term in the general solution of the associated homogeneous ODE multiply repeatedly by \( x \) until this is no longer the case.