# SET THEORY





$\approx \{1, 2, 3, 4, 5\}$


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# INTRODUCTION

Mathematics is often admired because one doesn't have to take anything on faith. Everything can be proved. But it's obvious that you must begin with some assumptions. Nothing can be proved out of nothing.

Sets were introduced towards the end of the nineteenth century and it was quickly realised that they could be a suitable foundation on which to build the whole of mathematics. The important thing about a set is that it is the embodiment of a property. For every property P we can form the set of all things having that property, that is $\{x \mid Px\}$, or so it was assumed. But the Russell paradox arises when Px means '$x \notin x$', for if $S = \{x \mid x \notin x\}$ then $S \in S$ if and only if $S \notin S$.

Axiomatic Set Theory was developed as a way of avoiding this pitfall, by legislating which properties can be allowed to give rise to sets. There were several ways proposed for doing this, all of which are equivalent to one another. In these notes we build upon the Zermelo-Fraenkel Axioms.

We show, assuming the **ZF axioms**, how we can develop a set theory in which everything we need is a set. Numbers can be defined as sets, ordered pairs can be defined as sets, functions can be defined as sets. Indeed every object in mathematics can be defined as a set.

Virtually every mathematical theorem can be proved by building on the ZF axioms. We illustrate the method by developing those parts of mathematics that normally use a lot of intuition, such as arithmetic, geometry and trigonometry.

Some mathematical theorems need to assume an additional axiom, the Axiom of Choice.

The startling thing is that the only set whose existence we assume in any absolute sense is the empty set. Various axioms allow us to construct a whole range of sets from the empty set, but there's a sense in which the whole of mathematics can be created from a void!

We develop the natural numbers as sets. For example the number 3 will be defined as:

$$\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}.$$

Well, at least it has 3 elements! Then we define the positive rational numbers, the positive real numbers, the set of all real numbers and finally the set of complex numbers – all as sets of sets of …

One of the principles we must follow is not to make any assumption based on geometric intuition. None of our proofs can be based on a diagram. This makes life difficult for us when it comes to trigonometry. Even the number $\pi$ has to be redefined. We can't define it in terms of the circumference of a circle. But fortunately we can still do it. We define the cosine function as a power series and define $\pi/2$ as the smallest positive real number whose cosine is zero (after establishing that such a thing exists).

The natural numbers are a way of counting finite sets, but for infinite sets we need to develop **infinite cardinal numbers**. The amazing thing is that there is more than one. Following the work of Georg Cantor we see that there are bigger and bigger infinite numbers. And eventually we see that even these infinite numbers are themselves sets, built up from the empty set.

Then we come across statements about sets that are unprovable, that is where it can be proved that there cannot be a proof that they are true, nor a proof that they are false. A famous, and useful, example is the **Axiom of Choice** and its equivalent formulation as **Zorn's Lemma**. This must be accepted, or rejected, as an article of faith. Or, in other words, if we want to use it we must accept it as an additional axiom.

The Axiom of Choice roughly says that if we have a set of non-empty sets there exists a set that consists of one element from each, or in other words it's possible to choose elements from a collection of non-empty sets, even if the collection is infinite. Put like this it seems plausible, yet it cannot be proved using the standard ZF axioms. We need the extra axiom – the Axiom of Choice. It seems reasonable to accept it because intuitively such choices are possible even with infinitely many infinite sets. However a consequence of the Axiom of Choice is that a solid ball can be decomposed into finitely many pieces and reassembled to form *two* solid balls of the same radius!  The ‘pieces’ are subsets but would be more like clouds than pieces, so disconnected that their volume cannot be defined, and so the law of conservation of volume is not broken.

Then we make a brief excursion into the theory of ordinal numbers, a generalisation of the familiar $1^{st}$, $2^{nd}$, $3^{rd}$, …

A study of Axiomatic Set Theory causes one to look at the nature of mathematical truth and takes one to bizarre corners of the mathematical universe where logic survives, but only just!  With a universe created from a void, axioms that must be taken on faith, and facts that are unknowable, axiomatic set theory sounds eerily like a religion. It is certainly a side of mathematics that is quite different to anything you have ever seen before!

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