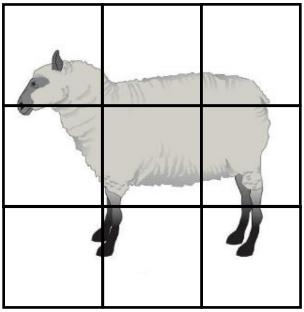
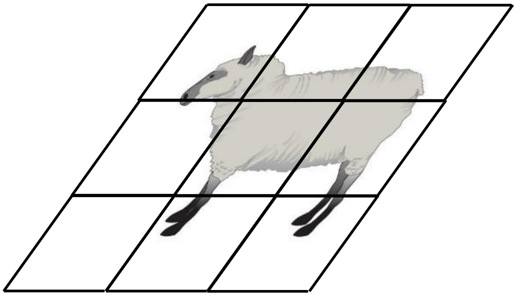
MATRICES



SHEEP



SHEARED SHEEP

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 & 0.3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

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These notes were prepared for students at Macquarie University in Australia but are freely available to anyone. However if you make use of them and are not a Macquarie University student it would be nice if you could email me at christopherdonaldcooper@gmail.com to let me know where you are from. And, if you are from outside of Australia perhaps you could send me a postcard of where you are from to pin up on my wall (Christopher Cooper, 31 Epping Avenue, EASTWOOD, NSW 2122, Australia).

INTRODUCTION

Linear Algebra is the study of vectors, matrices and vector spaces. A vector is simply a list of values considered as a single entity. The values, called the components of the vector, can be real numbers, or indeed they may come from one of the many systems that behave algebraically like the real numbers – the systems called fields. These lists can be written as a row or as a column, giving rise to row and column vectors respectively.

We begin with vectors with two components (x, y) such as are used to represent a point in a plane. The algebra of vectors is developed and the geometrical interpretation is heavily emphasised. Certain functions that take vectors to vectors, such as rotations, are called linear transformations because they take straight lines to straight lines. These need four numbers to describe them, arranged in a 2×2 table called a matrix.

A lot can be learnt about matrices very quickly by sticking to 2×2 matrices. These matrices are like generalised numbers that can be added, subtracted, multiplied and, in certain cases, divided, just like numbers. But there are important differences between the algebra of matrices and the algebra of numbers which we learnt at school.

There are two major differences. Firstly AB need not equal BA if A, B are matrices. As a consequence, many of the 'facts' we learnt in high school algebra don't work for matrices. We have to learn our algebra all over again!

The other major difference between the algebra of matrices and that of real numbers is the fact that it's possible to have AB = 0 in cases where neither A nor B is zero.

Having mastered 2×2 matrices we move on to 3 dimensions and 3×3 matrices and 3×3 determinants. A determinant is a single number extracted from a whole matrix that is useful in many applications. We emphasise the geometry behind determinants by defining 2×2 determinants as areas and 3×3 determinants as volumes.

Having become quite familiar with 2×2 and 3×3 matrices and determinants we move on to the general $n \times n$ case. Most of the matrix theory generalises readily to the $n \times n$ case but we need to define determinants in a non-geometric way since we have no intuitive concept of higher dimensional spaces. The general algebraic definition is shown to be compatible with the 2×2 and 3×3 cases and the properties of determinants are proved for the general case.

When n is bigger than 3 we have little use for $n \times n$ matrices geometrically. However one of their big roles lies in the analysis of systems of linear equations. Algorithms for solving such systems are presented and are analysed using matrices.

An important concept of matrix algebra is that of eigenvalues and eigenvectors. These are numbers and vectors that can be associated with a square matrix that prove to be extremely useful. They can be used in a process called diagonalisation, where most $n \times n$ matrices can be transformed to much simpler ones called ‘diagonal matrices’.

CONTENTS

6. EIGENVALUES AND EIGENVECTORS

7. DIAGONALISATION