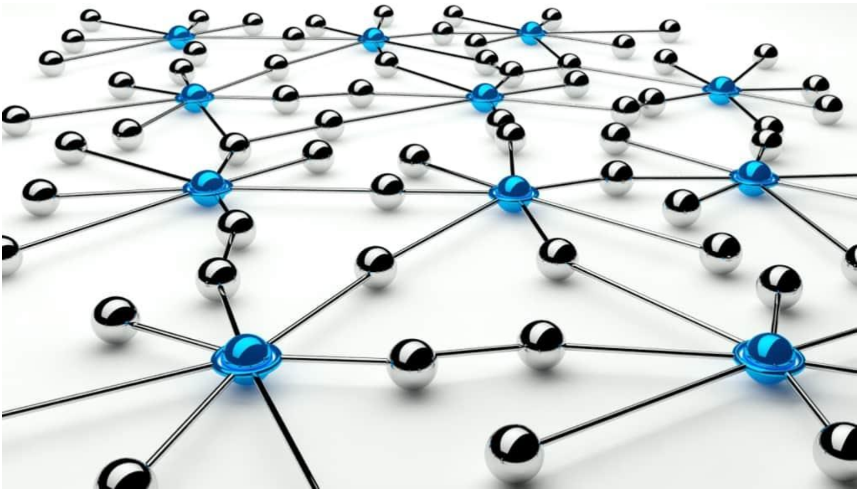
Graph Theory



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These notes were prepared for students at Macquarie University in Australia but are freely available to anyone. However if you make use of them and are not a Macquarie University student it would be nice if you could email me at christopherdonaldcooper@gmail.com to let me know where you are from. And, if you are from outside of Australia perhaps you could send me a postcard of where you are from to pin up on my wall (Christopher Cooper, 31 Epping Avenue, EASTWOOD, NSW 2122, Australia).

INTRODUCTION

When it's finished, this set of notes will provide a solid introduction to Graph Theory. These graphs aren't the familiar functions plotted on the *x*-*y* plane.

A **graph**, often called a **network**, is simply a set of objects called **vertices**, some or all of which are connected by other objects called **edges**. These terms come from the way we usually draw graphs, with the vertices being points in a plane, and the edges being curves, usually straight line segments, that connect one vertex to another.

The layout is irrelevant. The only feature of each edge that is important is which vertex is joined to which. Sometimes the edges are given a direction, but mostly we'll be playing with **undirected** graphs.

A **cycle** is a sequence of edges, each adjacent to the next, that returns to the starting point. Many graphs have no cycles. They are called **forests** because, when drawn, they look a bit like an abstract drawing of a real forest. A **tree** is amusingly defined as a 'connected forest'.

Other graphs do have cycles. The shortest cycle has three edges, and is called a **triangle**. One whole chapter is devoted to graphs that do or don't contain any triangles.

An **Eulerian cycle** is a cycle that passes through each vertex exactly once and a **Hamiltonian cycle** is one that passes along every edge exactly once. There are conditions that ensure that a graph has either one or both of these.

The **degree** of a vertex is the number of edges that come out from it. There are many things we can say about the list of degrees of the vertices of a graph.

We can **colour** a graph. This means colouring the vertices. We don’t need to use actual colours but can simply assign each vertex with a label that represents its colour. If the vertices represent classes in a school and the edges connect two classes that have at least one common student, then the colours might represent the time slots at which the classes can be held.

Here it’s important that two vertices that are **adjacent** (connected by an edge) must have different colours. For if two classes have overlapping students they must be assigned different time slots.

A certain graph can be ***n*-coloured** if the vertices can be coloured with *n* colours in such a way that adjacent vertices are given different colours. A related matter is **map colouring**.

A **weighted** connected graph is one where we assigned a number to each edge, called its **weight**. A **spanning tree** is one that leaves out certain edges but still reaches every vertex. A **minimal spanning tree** is one that has the smallest total weight. If the vertices represent towns, and the edges represent which towns we wish to connect by railroads, a minimal spanning tree might be a very desirable way to lay out our tracks.

CONTENTS

5. DEGREE PARTITIONS

6. GRAPHS ON SURFACES

7. GRAPH COLOURING

8. MINIMAL SPANNING TREES