# 8. INTEGRATION

## §8.1 The Area Function

The area under a graph can be a very useful tool but counting squares is far too crude a technique to use. Like differentiation there's a technique that can take us straight from the equation of a function to the equation of the corresponding area function without having to sketch the curve.

Suppose we fix $x = x_0$ as the starting point for working out the area under a graph. We define the area function, $A(x)$, to be the area between the curve, the $x$-axis, the vertical line $x = x_0$, and the vertical line at $x$, with $x \geq x_0$.
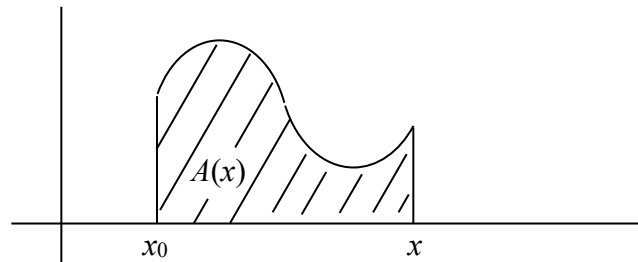


When $x = x_0$ the area is a "thin" strip with zero width and so the area $A(x_0)$ is zero. As $x$ increases the right hand boundary moves to the right, sweeping out more area and so $A(x)$ increases with $x$. (At least this is the case where the curve lies above the $x$-axis and areas are counted positive. Whenever the curve goes below the $x$-axis, the area below is interpreted as being negative and so $A(x)$ decreases as $x$ increases.)

In certain special cases we can work out the area function using simple geometry.

**Example 1:** Find the area function for the constant function $y = 3$, taking $x_0 = 0$ for the left-hand endpoint.
**Solution:** The graph of this function is a horizontal straight line and the area we have to measure is the area of the rectangle enclosed by the two axes, the line $y = 3$ and the vertical line at $x$. This area is clearly $3x$.
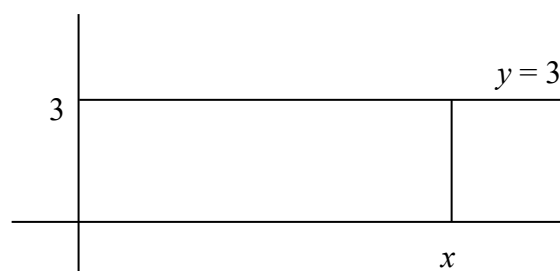


Now if we'd taken $x_0 = 1$ as the left hand endpoint the rectangle would have been shorter by 1 unit and so the area function would have been $3(x - 1) = 3x - 3$. The two areas differ by 3, and this is simply the area between $x = 0$ and $x = 1$ that was included the first time but left out the second time.

Clearly if we choose any other left-hand endpoint the area function will differ from $3x$ by some constant. For this reason we write the area function as $3x + c$ where $c$ is called an **arbitrary constant**. The constant $c$ ceases to become arbitrary once we settle on a specific left-hand endpoint.

## §8.2 The Indefinite Integral

What we've called the "area function" is more properly called the **indefinite integral**. It is called an "integral" because it collects together (integrates) all the strips of area that are generated as the right-hand boundary moves across. It's called "indefinite" because we use an "unspecified left-hand endpoint". (We could have had both endpoints being unspecified but that would have resulted in a function of two variables and this turns out to be unnecessary.)

Now the symbol that's used for the indefinite integral is rather unusual, and may seem strange when you first see it. We write

$$\int y \, dx$$

to represent the indefinite integral of y *with respect* to x. Just as with the $\frac{dy}{dx}$ notation for the derivative we incorporate the name of the independent variable. This allows us to change variables, a very useful technique of integration.

The rather strange symbol, $\int$ , is called the "integral sign". It looks rather like the letter $S$ that has somehow been stretched and that's where it comes from. Integration is very much like summation. If we divided the area under a graph into vertical strips the total area would be the *sum* of the areas of each of these strips. Writing the word Sum as $\int$um we can see the connection.

Just remember that integrals are simply area functions under a more sophisticated name. In the example above we showed, by simple geometry, that $\int 3 \, dx = 3x + c$. We can use the formula for the area of a triangle to integrate functions of the form $y = ax + b$.

**Example 2:** Find $\int x + 3 \, dx$ .

**Solution:** The equation $y = x + 3$ is the equation of a straight line. Taking, for convenience, the left-hand endpoint $x_0 = 0$, the required area is the area of a *trapezium*. (A "trapezium" is a quadrilateral, a figure with four straight sides, in which two sides are parallel.) You may or may not know the formula for the area of a trapezium, but the easiest way to work it out is to split it into a rectangle and a triangle.
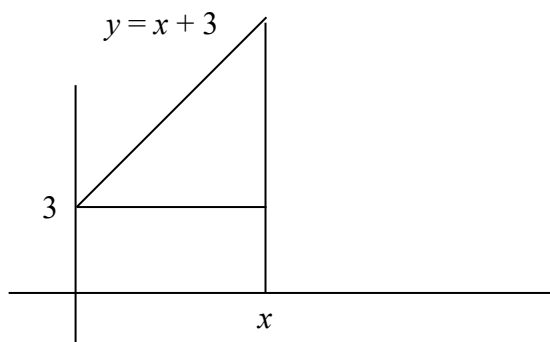


The area of the rectangle is $3x$ and the area of the triangle is ½ base times perpendicular height, which is ½ $x^2$.

So $\int x + 3 \, dx = \frac{1}{2} x^2 + 3x + c$.

Don't forget the $+ c$ with indefinite integrals. We chose a left-hand endpoint of $x = 0$ for convenience but it might have been any other value. The various answers will all be essentially the same, just differing in the constant term.

The next function we might wish to integrate is a quadratic, like $y = x^2$. Here our simple geometry lets us down and we need an insight into the deep connection between differentiation and integration. This is known as the **Fundamental Theorem of Calculus**.

## §8.3 The Fundamental Theorem of Calculus

Differential Calculus is all about slopes and Integral Calculus is all about areas and you might not think that slopes and areas have much to do with each other, apart from being different aspects of a graph. But the surprising fact is that these are *inverse operations*.

An example of a pair of inverse operations with which you are familiar is $x^2$ and $\sqrt{x}$. Squaring and taking the square root are inverse, or reverse, with respect to each other. They are like going one way or the other down a two-way street. If you square a positive number, and then take the square root you get back to the number you started with. And if you square the square root you also recover the original number. It doesn't matter which one you do first. If you then do the other you're back where you started.

For example squaring 9 we get 81 and taking the square root of 81 we get 9. If we take the square root of 9 we get 3 and squaring this square root we again get 9.

There are plenty of other pairs of inverse operations in mathematics. Halving and doubling, subtracting 1 and adding 1 are just two more familiar examples. Well, differentiation and integration are simply inverse to one another.

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**The derivative of the integral of $f(x)$ is $f(x)$ itself.**

$$\text{i.e. } \frac{d\int f(x)\,dx}{dx} = f(x)$$

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To get some idea of why this is so (this doesn't attempt to be a fully rigorous proof) let's suppose we have the graph of $y = f(x)$ and let $A = \int f(x)\,dx$ be the area under the curve up to $x$ (from some starting value $x = x_0$).

Let $x$ change by an increment $\Delta x$ and let the corresponding increments to $y$ and $A$ be $\Delta y$ and $\Delta A$. The new area, up to $x + \Delta x$, is $A + \Delta A$ and so $\Delta A$ represents the extra area, between $x$ and $x + \Delta x$.
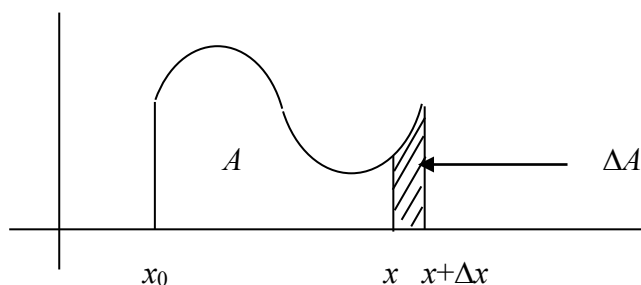


The extra bit of area is roughly a trapezium (if $\Delta x$ is small the small portion of curve can be approximated by a straight line).
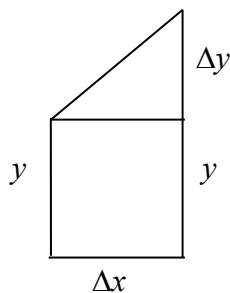


The area of this trapezium is $y\Delta x$ (the rectangle) plus $\frac{1}{2} \Delta x . \Delta y$ (the triangle).

So $\Delta A \approx y.\Delta x + \frac{1}{2} \Delta y.\Delta x$ and so $\dfrac{\Delta A}{\Delta x} \approx y + \frac{1}{2} \Delta y$. (The symbol "$\approx$" here means "is approximately equal to".)

Now as $\Delta x \to 0$, $\Delta y \to 0$ and $\dfrac{\Delta A}{\Delta x} \to \dfrac{dA}{dx}$. Hence $\dfrac{dA}{dx} \approx y$. It seems quite plausible, that the approximation becomes an exact equality in the limit and so $\dfrac{dA}{dx} = y = f(x)$. This is not just plausible, it is in fact perfectly true, which a more rigorous argument could show. We content ourselves with just accepting this fact. Since $A = \int f(x)\, dx$ we've proved (subject to the bit about the approximation becoming exact in the limit) the Fundamental Theorem.

We won't prove the reverse direction, that if you integrate the derivative you get back to where you started. For a start it's not quite true, in so far as the integral of the derivative has an arbitrary constant, which the original function does not. Secondly, the direction we've proved is the one we'll need.

# §8.4 Integration as Anti-Differentiation
How do you find the square root of a number? "Press the square root button on my calculator." But what if you had to do it by hand? Without a calculator squaring a number might be hard going, but finding the square root would be impossible. Or would it?

We could use the fact that the square root squared is the original number to get better and better approximations. Suppose we wanted to find $\sqrt{2}$.

Well $1^2 = 1$ and $2^2 = 4$. Since 2 lies between 1 and 4 it's clear that $\sqrt{2}$ lies between 1 and 2. Let's try halfway. Since $1.5^2 = 2.25$ we've overshot the mark, but not by much. Let's try $1.4^2$. That's 1.96 which is very close, but just a shade too small. Let's try 1.41. Squaring we get $1.41^2 = 1.9881$. A bit more. $1.42^2 = 2.0164$. So $\sqrt{2}$ is 1.41 something. By continuing in this way we can get $\sqrt{2} = 1.41421356237\ldots$as accurately as we want.

Notice that we found the square root, or at least got a good approximation to it, by squaring. We used the inverse operation of squaring, trying various possibilities, till we got one that works, or nearly so.

We can do the same with integration. If you want $\int x^2\, dx$ all you have to do is find something which gives $x^2$ when you differentiate it. Unlike the square root example we can find the integral exactly but, like that example, we try various things and adjust our guess by a little piece of reverse engineering.

**Example 3: Find $\int x^2\, dx$ by "reverse engineering".**
What gives $x^2$ when we differentiate it? Well remember that when we differentiate a power of $x$ the power goes down by 1. Something else happens to the coefficient but we'll worry about that later. So if the power goes down by 1 on differentiation it should go up by 1 when we integrate. So let's try $x^3$. If we differentiate $x^3$ we get $3x^2$. Not bad, but it's 3 times too big. No worries. We simply divide by 3 and try $\dfrac{x^3}{3}$. Differentiating that we get exactly $x^2$.

So $\int x^2\, dx = \dfrac{x^3}{3}$.

But is it? If we differentiate $\dfrac{x^3}{3} + 1$ we still get $x^2$. Doesn't this mean that $\int x^2\, dx = \dfrac{x^3}{3} + 1$?

Well, yes. If we add any constant to $\dfrac{x^3}{3}$, and then differentiate, we get $x^2$. So what's going on here?

We said that $\int f(x)\,dx$ is *that function* which when differentiated gives $f(x)$. But it seems there are *many* such functions. So which is the correct answer? The correct answer is that they're *all* correct. In other words the answer isn't unique. Remember the arbitrary constant?

If $g(x)$ and $h(x)$ are two functions whose derivative is $f(x)$ then

$$\frac{d[g(x) - h(x)]}{dx} = f(x) - f(x) = 0.$$

Now the only functions whose derivative is zero at every point are those that run horizontally in a straight line, the constant functions. So $g(x)$ and $h(x)$ differ by a constant and so the answer to the integral is unique apart from the existence of an arbitrary constant.

So to integrate a function you just make an educated guess. Then you differentiate to see if it works. If it doesn't you make suitable modifications to your guess and try again. Once you find something whose derivative is what you want you write it down as your answer and you *add a "+c" term to it*. **Never forget the "+c" with indefinite integrals.**

## Example 4: Find $\int(10x^4 + 3)\,dx$

**Solution:** We want an $x^4$ so try $x^5$. The derivative of $x^5$ is $5x^4$. So if we take $2x^5$ we'll get the $10x^4$ term when we differentiate. To get the "3" we'll need a $3x$ term. So our modified guess is $2x^5 + 3x$. The derivative of this is $10x^4 + 3$. So our answer is $\int(10x^4 + 3)\,dx = 2x^5 + 3x + c$.

But if each of $\int 10x^4\,dx$ and $\int 3\,dx$ have their own arbitrary constant, shouldn't we have

two arbitrary constants for the sum and write $\int(10x^4 + 3)\,dx = 2x^5 + c_1 + 3x + c_2$. The answer is

that we can if we want to. But the sum of two arbitrary constant is a single arbitrary constant (we could write $c = c_1 + c_2$) and so for simplicity we bundle all the arbitrary constants into one.

## §8.5 The Power Formula

We obviously don't want to have to always use trial and error if we can help it. Can't we find a formula for $\int x^n\,dx$? Of course we can. But we'll use our trial-and-error method in order to develop that formula.

We want to find something that gives us $x^n$ when we differentiate. Clearly $x^{n+1}$ is a good place to start. Now the derivative of $x^{n+1}$ is $(n+1)x^n$, the right power but the wrong coefficient. No problem. Just divide by $n+1$ to get $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$.

There's just one teeny, teeny problem. What if $n = -1$? Whenever you divide, a little alarm should go off in your head. Could you be dividing by zero? If you divide by a specific constant you'll know very well whether it's zero. But expressions may not look very much like zero and yet could be zero for certain values of the variables.

Since $x^{-1} = \frac{1}{x}$ we've struck a problem with $\int \frac{1}{x}\,dx$. It seems we can't find anything which

gives this when differentiated. Now there *is* an answer to $\int \frac{1}{x}\,dx$, but it's quite a different sort of

function to the powers of $x$ that we've been dealing with. This exception will have to wait till later.

$$\boxed{\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c \quad \text{provided } n \neq -1}$$

The easiest way to remember this is to remember it as "the power goes up by 1 and you divide by this increased power".

## §8.6 Sums and Constant Multiples

In the example above we found the integrals of $10x^4 + 3$ by integrating $10x^4$ and 3 separately. In general we can integrate a function term by term.

$$\int [f(x) + g(x)]\ dx\ =\ \int f(x)\ dx\ +\ \int g(x)\ dx$$

The reason for this is that the same property holds for derivatives. If the indefinite integrals of $f(x)$ and $g(x)$ are $F(x)$ and $G(x)$ respectively then $\dfrac{d[F(x) + G(x)]}{dx} = \dfrac{dF(x)}{dx} + \dfrac{dG(x)}{dx} = f(x) + g(x)$.

Looking at this in reverse, $\int (f(x) + g(x))\ dx\ = F(x) + G(x) = \int f(x)\ dx\ +\ \int g(x)\ dx$ .

Something similar applies to constant multiples.

$$\int k\, f(x)\ dx\ =\ k\int f(x)\ dx$$

The reason is that if $\int f(x)\ dx = F(x)$ and $y = F(x)$ then $\dfrac{dy}{dx} = f(x)$ and so $\dfrac{d(ky)}{dx} = kf(x)$. Viewing this in reverse we have $\int k\, f(x)\ dx = kF(x) = k\int f(x)\ dx$ .

In this way we can integrate an polynomial-like functions, as long as there are no $\dfrac{1}{x}$ terms.

**Example 5:** Find $\int (18x^5 + 8x^3 + 2)\ dx$ .

**Solution:** $\int (18x^5 + 8x^3 + 2)\ dx\ = 3x^6 + 2x^4 + 2x + c.$

**Example 6:** Find $\displaystyle\int (3\sqrt{x} + \frac{4}{\sqrt{x}})\ dx$ .

**Solution:** $\displaystyle\int (3\sqrt{x} + \frac{4}{\sqrt{x}})\ dx\ = \int (3x^{1/2} + 4x^{-1/2})\ dx$

$$= \frac{3x^{3/2}}{3/2} + \frac{4x^{1/2}}{1/2} + c$$
$$= 2x^{3/2} + 8x^{1/2} + c$$
$$= 2x\sqrt{x} + 8\sqrt{x} + c.$$

## §8.7 Definite Integrals

The indefinite integral is the area function, giving the area under the curve from some fixed, but unspecified, starting point $x_0$, up to a varying $x$. But often we want a specific area, say the area from $x = a$ to $x = b$. This is a **definite integral** because the answer is a number, not a function of $x$. Of course it depends on $a$ and $b$ but we're usually given specific values for these. We write this definite integral as $\displaystyle\int_a^b f(x)\ dx$. Notice the **limits of integration** are written at the top and bottom of the integral sign.

$A = \displaystyle\int_a^b y\ dx$ **is the area between** $y = f(x)$ **and the** $x$**-axis, from** $x = a$ **to** $x = b$

$y = f(x)$

x-axis

a          b

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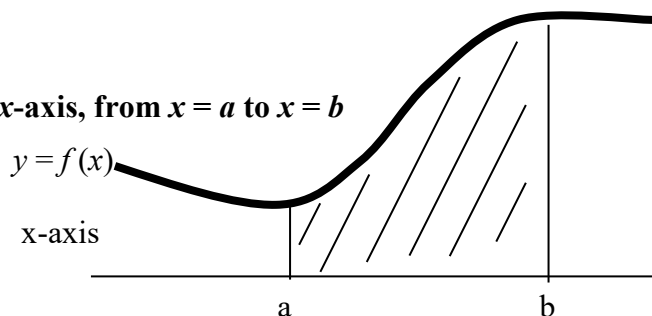
Let $F[x] = \int f(x)\,dx$ be the area function, giving the area from some $x_0$ to $x$. Then $F[a]$ is the area from $x = x_0$ to $x = a$ and $F[b]$ is the area from $x = x_0$ to $x = b$. Irrespective of what $x_0$ is it's clear that the area from $x = a$ to $x = b$ is the difference between the two, that is $\int_a^b f(x)\,dx = F[b] - F[a]$.

We have a special notation to denote such expressions. We write $[F(x)]_a^b$ to mean $F(b) - F(a)$. So:

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**If $F(x)$ is the indefinite integral of $f(x)$ with respect to $x$:**
$$\int_a^b f(x)\,dx = [F(x)]_a^b = F[b] - F[a]$$

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**Example 7:** Find the area between $y = x^2$ and the $x$-axis from $x = 1$ to $x = 2$.
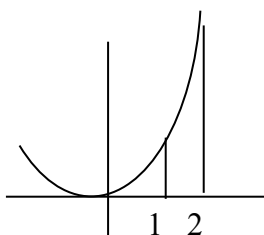


**Solution:**

The area $= \int_1^2 x^2\,dx = \left[\dfrac{x^3}{3}\right]_1^2 = \dfrac{8}{3} - \dfrac{1}{3} = \dfrac{7}{3}$.

Notice that we didn't bother with the "$+ c$" term in the indefinite integral. That's because we'd get two "$+ c$" terms which would cancel: $\int_1^2 x^2\,dx = \left[\dfrac{x^3}{3} + c\right]_1^2 = \left(\dfrac{8}{3} + c\right) - \left(\dfrac{1}{3} + c\right) = \dfrac{7}{3}$.

## §8.8 Linear Substitutions

Just as there are rules for differentiation, such as the product rule and the chain rule, there are rules for integration. However these are somewhat more difficult and we omit all but one of them, the Linear Substitution Rule.

But before we discuss this rule it should be pointed out that while the rules for differentiation are sufficient to enable us to differentiate any function we can dream up, no matter how complicated, such is not the case with integration. There are fairly simple functions whose integrals can't be expressed in terms of simple functions. New functions need to be invented for them. A famous example of an integral that "can't be done" is $\int e^{-x^2}\,dx$, an integral much used in statistics..

For now we'll concentrate on the one simple rule for integration that will concern us – the Linear Substitution Rule. Suppose we have an integral of the form $\int f(ax + b)\,dx$, where $a, b$ are constants. Suppose also that we are able to integrate the function $f(x)$. For example we might want to find $\int (3x + 2)^5\,dx$. Here $a = 3$ and $b = 2$ and the function is $f(x) = x^5$.

We know that $\int x^5\,dx = \dfrac{x^6}{6} + c$ so $\int (3x + 2)^5\,dx$ should look something like $\dfrac{(3x + 2)^6}{6}$.

It does, and in fact $\int(3x+2)^5\,dx = \dfrac{1}{3}\cdot\dfrac{(3x+2)^6}{6} + c = \dfrac{(3x+2)^6}{18} + c.$ In effect we

substitute $u = 3x + 2$ and find $\int u^5\,du = \dfrac{u^6}{6} + c.$ But, as is always the case with substitutions, we need to make an adjustment. If we were using the chain rule to differentiate, the substitution $u = 3x + 2$ would require us to multiply by $\dfrac{du}{dx} = 3.$ With integration we *divide* by 3 instead.

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**The Linear Substitution Rule**
**If $\int f(x)\,dx = F(x)$ and $a \neq 0$ then**

$$\int f(ax+b)\,dx = \frac{1}{a}\,F(ax+b).$$

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**Why is it so?**

If we differentiate $y = \dfrac{1}{a}\,F(ax+b)$ with respect to $x$, by the Chain Rule, we let $u = ax + b$ and so

$y = \dfrac{1}{a}\,F(u).$ Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = \dfrac{1}{a}\dfrac{dF(u)}{du}\cdot a = \dfrac{dF(u)}{du} = f(u) = f(ax+b).$ Hence $y = \int f(ax+b)\,dx.$

**Example 8:** Find $\int(4x-7)^{11}\,dx.$

**Solution:** Let $u = 4x - 7.$ Then $\int(4x-7)^{11}\,dx = \dfrac{1}{4}\int u^{11}\,du = \dfrac{1}{4}\cdot\dfrac{u^{12}}{12} + c = \dfrac{1}{48}(4x-7)^{12} + c.$

**Example 9:** Find $\int(4x-3)^2\,dx.$

**Solution:** Let $u = 4x - 3.$ Then $\int(4x-3)^2\,dx = \dfrac{1}{4}\int u^2\,du = \dfrac{1}{4}\cdot\dfrac{u^3}{3} + c = \dfrac{1}{12}(4x-3)^3 + c.$

In this case we could have expanded $(4x-3)^2$ and integrated in the usual way. Let's see if we get the same answer.

$\int(4x-3)^2\,dx = \int(16x^2 - 24x + 9)\,dx = \dfrac{16x^3}{3} - 12x^2 + 9x + c.$

Are these answers the same? Well, yes and no. If we were to take the trouble to expand our first answer $\dfrac{1}{12}(4x-3)^3$ we'd get $\dfrac{16x^3}{3} - 12x^2 + 9x - \dfrac{27}{12}.$ This is not quite the same as

$\dfrac{16x^3}{3} - 12x^2 + 9x,$ but the two answers differ by a constant. It's just that the constants are different in each case.

When you integrate a particular function by different methods your answers, ignoring the "+ c" term, needn't be the same. However they must only differ by a constant.

In Example 8 we could have found $\int(4x-7)^{11}\,dx$ by expanding $(4x-7)^{11}$ though clearly using linear substitution is far simpler. But there are many other cases where linear substitution is the only way.

In chapter 10 we'll be introducing some new functions, the logarithm function and the exponential function. Never mind at this stage what they mean. It's enough to know, for now, that the log function is written as **log** $x$ and the exponential function is $e^x.$ This last one is simply a certain number, called e, raised to the $x$'th power. The number $e$ is approximately 2.71828 but, like

π, it's decimal expansion goes on forever with no discernable pattern. Why use such a funny number? You'll have to wait till chapter 10.

But we can report to you now that if $y = \log x$ then $\dfrac{dy}{dx} = \dfrac{1}{x}$ and that if $y = e^x$ then $\dfrac{dy}{dx} = e^x$. Both of these results are interesting.

Remember that when we were finding $\int x^n \, dx$ we found that the answer was $\dfrac{x^{n+1}}{n+1} + c$ EXCEPT in the case $n = -1$ when this formula would give nonsense. So we had not been able to find $\int \dfrac{dx}{x}$, by which we mean $\int \dfrac{1}{x} \, dx$. Now we have an answer. If the derivative of $\log x$ is $\dfrac{1}{x}$ then by the Fundamental Theorem of Calculus, the integral of $\dfrac{1}{x}$ is $\log x$. We should perhaps qualify this. The log function doesn't exist if $x < 0$ and so if we integrate $\dfrac{1}{x}$ over a range of negative values of $x$ we won't get $\log x$. But $\int \dfrac{dx}{x} = \log x + c$ provided we're integrating over a range of positive values of $x$.

The fact that the derivative of $e^x$ is itself means that the integral of $e^x$ is $e^x$. This is a member of a family of functions that don't change when they're differentiated or integrated.

$$\boxed{\begin{array}{c} \textbf{Special Integrals} \\[6pt] \displaystyle \int \frac{dx}{x} = \log x + c \text{ for } x > 0 \\[10pt] \displaystyle \int e^x \, dx = e^x + c. \end{array}}$$

**Example 10:** Find $\int \dfrac{dx}{3 - x}$.

**Solution:** Let $u = 3 - x$. Then $\int \dfrac{dx}{3 - x} = \dfrac{1}{-1} \int \dfrac{du}{u} = - \int \dfrac{du}{u} = - \log u + c = - \log(3 - x) + c.$

**Example 11:** Find $\int e^{2x + 1} \, dx$.

**Solution:** Let $u = 2x + 1$. Then $\int e^{2x + 1} \, dx = \dfrac{1}{2} \int e^u \, du = \dfrac{1}{2} e^u + c = \dfrac{1}{2} e^{2x+1} + c.$

A substitution of the form $u = ax + b$ is called a *linear* substitution since the graph of the function $u = ax + b$ is a straight line with slope $a$. Note that the above rule doesn't extend to substitutions in general. For example you might try to find $\int e^{-x^2} \, dx$ by substituting $u = -x^2$. This might lead you to think that, since the derivative of $-x^2$ is $-2x$, then $\int e^{-x^2} \, dx = \dfrac{1}{-2x} \int e^u \, du$ but this is badly wrong. You must divide by the derivative BEFORE you integrate.

**NOTE:** There is a more general rule for Integration by Substitution but in this example it would merely transform one impossible integral into another.

# EXERCISES FOR CHAPTER 8

**Exercise 1:** In each of the following cases find $\int y \, dx$ .

    (i) $y = x^4$;
    (ii) $y = 2x^5 - 3x$;
    (iii) $y = 12x^5 + 4x + 4$;
    (iv) $y = 4x^5 - x^3 + 2$;
    (v) $y = (x + 1)^2$;
    (vi) $y = x + \sqrt{x}$ ;
    (vii) $y = x\sqrt{x}$ ;
    (viii) $y = (x + \sqrt{x})^2$;

    (ix) $y = \dfrac{4}{x^3}$ ;

    (x) $y = \dfrac{1 + x^2}{x^2}$ .

**Exercise 2:** Find $\displaystyle\int_1^2 y \, dx$ for each of the above functions (i) to (v).

**Exercise 3:** Find $\displaystyle\int_1^4 y \, dx$ for each of the above functions (vi) to (x).

**Exercise 4:** Use linear substitution to find the following indefinite integrals.

    (i) $\int (7x + 2)^3 \, dx$ ;

    (ii) $\int \dfrac{dx}{3x + 5} \, dx$ ;

    (iii) $\int e^{x+1} \, dx$ .

# SOLUTIONS FOR CHAPTER 8

**Exercise 1:**

(i) $\int x^4 \, dx = \frac{1}{5} x^5 + c$;

(ii) $\int 2x^5 - 3x \, dx = \frac{x^6}{3} - \frac{3x^2}{2} + c$;

(iii) $\int 12x^5 + 4x + 4 \, dx = 2x^6 + 2x^2 + 4x + c$;

(iv) $\int 4x^5 - x^3 + 2 \, dx = \frac{2x^6}{3} - \frac{x^4}{4} + 2x + c$;

(v) $y = (x + 1)^2 = x^2 + 2x + 1$, so $\int (x + 1)^2 \, dx = \frac{x^3}{3} + x^2 + x + c$;

(vi) $\int (x + \sqrt{x}) \, dx = \frac{x^2}{2} + \frac{2}{3} x^{3/2} + c = \frac{x^2}{2} + \frac{2}{3} x\sqrt{x} + c$;

(vii) $y = x\sqrt{x} = x^{3/2}$ so $\int x\sqrt{x} \, dx = \frac{2}{5} x^{5/2} + c$;

(viii) $y = (x + \sqrt{x})^2 = x^2 + 2x\sqrt{x} + x = x^2 + 2x^{3/2} + x$, so $\int (x + \sqrt{x})^2 \, dx = \frac{x^3}{3} + \frac{4}{5} x^{5/2} + \frac{x^2}{2} + c$;

(ix) $y = \frac{4}{x^3} = 4x^{-3}$ so $\int \frac{4}{x^3} \, dx = -2x^{-2} + c = -\frac{2}{x^2} + c$;

(x) $y = \frac{1 + x^2}{x^2} = \frac{1}{x^2} + 1 = x^{-2} + 1$ so $\int \left(\frac{1 + x^2}{x^2}\right) dx = -x^{-1} + x + c = -\frac{1}{x} + x + c$.

**Exercise 2:**

(i) $\int_1^2 x^4 \, dx = \left[\frac{1}{5} x^5\right]_1^2 = \frac{32 - 1}{5} = \frac{31}{5}$;

(ii) $\int_1^2 (2x^5 - 3x) \, dx = \left[\frac{x^6}{3} - \frac{3x^2}{2}\right]_1^2 = \left(\frac{64}{3} - \frac{12}{2}\right) - \left(\frac{1}{3} - \frac{3}{2}\right) = \frac{33}{2}$;

(iii) $\int_1^2 (12x^5 + 4x + 4) \, dx = [2x^6 + 2x^2 + 4x]_1^2 = (128 + 8 + 8) - (2 + 2 + 4) = 136$;

(iv) $\int_1^2 (4x^5 - x^3 + 2) \, dx = \left[\frac{2x^6}{3} - \frac{x^4}{4} + 2x\right]_1^2 = \left(\frac{128}{3} - 4 + 4\right) - \left(\frac{2}{3} - \frac{1}{4} + 2\right) = \frac{161}{4}$;

(v) $\int_1^2 (x + 1)^2 \, dx = \left[\frac{(x + 1)^3}{3}\right]_1^2 = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$.

In this case it would not have been difficult to expand $(x + 1)^2$ and avoided the need for substitution:

$\int_1^2 (x + 1)^2 \, dx = \int_1^2 (x^2 + 2x + 1) \, dx = \left[\frac{x^3}{3} + x^2 + x\right]_1^2 = \left(\frac{8}{3} + 4 + 2\right) - \left(\frac{1}{3} + 1 + 1\right) = \frac{19}{3}$.

**Exercise 3:**

(vi) $\int_1^4 (x + \sqrt{x})\, dx = \left[ \dfrac{x^2}{2} + \dfrac{2}{3} x\sqrt{x} \right]_1^4 = \left( 4 + \dfrac{16}{3} \right) - \left( \dfrac{1}{2} + \dfrac{2}{3} \right) = \dfrac{49}{6}$ ;

(vii) $\int_1^4 x\sqrt{x}\, dx = \left[ \dfrac{2}{5} x^{5/2} \right]_1^4 = \left( \dfrac{64}{5} \right) - \left( \dfrac{2}{5} \right) = \dfrac{62}{5}$ ;

(viii) $\int_1^4 (x + \sqrt{x})^2\, dx = \left[ \dfrac{x^3}{3} + \dfrac{4}{5} x^{5/2} + \dfrac{x^2}{2} \right]_1^4 = \left( \dfrac{64}{3} + \dfrac{128}{5} + 8 \right) - \left( \dfrac{1}{3} + \dfrac{4}{5} + \dfrac{1}{2} \right) = \dfrac{533}{10}$ ;

(ix) $\int_1^4 \dfrac{4}{x^3}\, dx = \left[ -\dfrac{2}{x^2} \right]_1^4 = \left( -\dfrac{2}{16} \right) - (-2) = \dfrac{15}{8}$ ;

(x) $\int_1^4 \left( \dfrac{1 + x^2}{x^2} \right) dx = \left[ -\dfrac{1}{x} + x \right]_1^4 = \left( -\dfrac{1}{4} + 4 \right) - (-1 + 1) = \dfrac{15}{4}$ .

**Exercise 4:**

(i) Let $u = 7x + 2$. Then $\int (7x + 2)^3\, dx = \dfrac{1}{7} \int u^3\, du = \dfrac{1}{7} \dfrac{u^4}{4} + c = \dfrac{(7x + 2)^4}{28} + c.$

(ii) Let $u = 3x + 5$. Then $\int \dfrac{dx}{3x + 5} = \dfrac{1}{3} \int \dfrac{du}{u} = \dfrac{1}{3} \log u + c = \dfrac{1}{3} \log(3x + 5) + c.$

(iii) Let $u = x + 1$. Then $\int e^{x+1}\, dx = \int e^u\, du = e^u + c = e^{x+1} + c.$